

THE CLASSIFICATION OF ALMOST SIMPLE $\frac{3}{2}$ -TRANSITIVE GROUPS

JOHN BAMBERG, MICHAEL GIUDICI, MARTIN W. LIEBECK, CHERYL E. PRAEGER, AND JAN SAXL

ABSTRACT. A finite transitive permutation group is said to be $\frac{3}{2}$ -transitive if all the nontrivial orbits of a point stabiliser have the same size greater than 1. Examples include the 2-transitive groups, Frobenius groups and several other less obvious ones. We prove that $\frac{3}{2}$ -transitive groups are either affine or almost simple, and classify the latter. One of the main steps in the proof is an arithmetic result on the subdegrees of groups of Lie type in characteristic p : with some explicitly listed exceptions, every primitive action of such a group is either 2-transitive, or has a subdegree divisible by p .

1. INTRODUCTION

Burnside, in his 1897 book [14, p 192, Theorem IX], investigated the structure of finite 2-transitive permutation groups. He proved that any such group is either affine or almost simple; in other words, the group has a unique minimal normal subgroup which is either elementary abelian and regular, or nonabelian simple. A transitive permutation group G on a set Ω is said to be $\frac{3}{2}$ -transitive if all orbits of G_α on $\Omega \setminus \{\alpha\}$ have the same size, with this size being greater than 1. For convenience, we also count S_2 as a $\frac{3}{2}$ -transitive group. A nontrivial, nonregular, normal subgroup of a 2-transitive group has to be $\frac{3}{2}$ -transitive. The term $\frac{3}{2}$ -transitive was first used in Wielandt's book [68], where he extended Burnside's proof to show that any $\frac{3}{2}$ -transitive group is either primitive or a Frobenius group (that is, every two point stabiliser is trivial). The classification of 2-transitive groups is a notable consequence of the classification of finite simple groups (see [23, 28, 31, 40, 52], and [15] for an overview).

In this paper, we obtain two results along the way towards the classification of $\frac{3}{2}$ -transitive groups. First, in parallel to Burnside's structure theorem, we prove

Theorem 1.1. *Every finite primitive $\frac{3}{2}$ -transitive group is either affine or almost simple.*

Our second result deals with the almost simple case. A *subdegree* of a transitive permutation group is a size of an orbit of the point stabiliser.

Theorem 1.2. *Let G be a finite almost simple $\frac{3}{2}$ -transitive group of degree n on a set Ω . Then one of the following holds:*

- (i) G is 2-transitive on Ω .
- (ii) $n = 21$ and G is A_7 or S_7 acting on the set of pairs of elements of $\{1, \dots, 7\}$; the size of the nontrivial subdegrees is 10.
- (iii) $n = \frac{1}{2}q(q-1)$ where $q = 2^f \geq 8$, and either $G = \text{PSL}_2(q)$ or $G = \text{P}\Gamma\text{L}_2(q)$ with f prime; the size of the nontrivial subdegrees is $q+1$ or $f(q+1)$, respectively.

Theorem 1.2 is a combination of Theorems 4.1, 5.1(B), 7.2 and 8.1. The examples in parts (ii) and (iii) can be found in Lemmas 4.3 and 6.2. The groups in (iii) were first investigated in the context of $\frac{3}{2}$ -transitivity by McDermott [53], and Camina and McDermott [17]. A characterisation of the groups in (i) and (iii) as the only $\frac{3}{2}$ -transitive groups with trivial Fitting subgroup and all two-point stabilisers conjugate was given by Zieschang [69].

Affine $\frac{3}{2}$ -transitive groups will be the subject of a future paper. The soluble case was handled completely by Passman in [60, 61, 62].

Our proof of Theorem 1.2, in the main case where G has socle of Lie type, uses the following result concerning the arithmetic nature of subdegrees of such groups.

Theorem 1.3. *Let G be an almost simple group with socle L of Lie type of characteristic p . Let G act primitively on a set Ω , and let H be the stabiliser of a point. Assume that p divides $|H|$. Then one of the following holds:*

2000 *Mathematics Subject Classification.* Primary 20B05, 20B15.

This paper forms part of an Australian Research Council Discovery Project. The second author was supported by an Australian Research Fellowship while the fourth author was supported by a Federation Fellowship of the Australian Research Council.

TABLE 1. 2-transitive cases for Theorem 1.3

L	$ \Omega $	Comment(s)
$\text{PSL}_2(4)$	6	
$\text{Sp}_4(2)'$	6	
$\text{Sp}_4(2)'$	10	
$\text{PSL}_2(9)$	6	
$\text{PSL}_4(2)$	8	
$\text{PSL}_3(2)$	8	$G = L.2$
$G_2(2)'$	28	
${}^2G_2(3)'$	9	$G = L.3$

TABLE 2. Other exceptions to Theorem 1.3

L	$ \Omega $	H	Subdegrees	Comment(s)
$\text{PSU}_3(5)$	50	$N_G(A_7)$	1, 7, 42	
$\text{PSp}_4(3)$	27	$N_G(2^4.A_5)$	1, 10, 16	
$G_2(2)'$	36	$\text{PSL}_3(2)$	1, 7, 7, 21	$G = L$

- (i) G has a subdegree divisible by p .
- (ii) G is 2-transitive on Ω : here either $G = \text{Sp}_{2d}(2)$ for $d \geq 3$ with $H = O_{2d}^\pm(2)$ and $|\Omega| = 2^{d-1}(2^d \mp 1)$, or G is detailed in Table 1.
- (iii) $L = \text{PSL}_2(q)$ with $q = 2^f \geq 8$, $|\Omega| = \frac{1}{2}q(q-1)$, $H \cap L = D_{2(q+1)}$, and $|G:L|$ is odd.
- (iv) G is detailed in Table 2.

We remark that the examples in Table 1 and lines 2 and 3 of Table 2 are in some sense degenerate as they arise due to exceptional isomorphisms with either alternating groups or groups of Lie type of different characteristic, namely $\text{PSL}_2(4) \cong \text{PSL}_2(5)$, $\text{Sp}_4(2)' \cong \text{PSL}_2(9) \cong A_6$, $\text{PSL}_4(2) \cong A_8$, $\text{PSL}_3(2) \cong \text{PSL}_2(7)$, $G_2(2)' \cong \text{PSU}_3(3)$, ${}^2G_2(3)' \cong \text{PSL}_2(8)$ and $\text{PSp}_4(3) \cong \text{PSU}_4(2)$. The conditions $G = L.2$ and $L.3$ in lines 6 and 8 of Table 1 are required so that p divides $|H|$.

Theorem 1.3 follows from Theorems 5.1(A) and 7.1. Theorem 3.4 in Section 3 is a result of a similar flavour for general primitive permutation groups.

We shall obtain several consequences of the above results. The first is immediate from Theorems 1.1 and 1.2.

Corollary 1.4. *The socle of a primitive $\frac{3}{2}$ -transitive group is either regular or $\frac{3}{2}$ -transitive.*

The orbitals of $\frac{3}{2}$ -transitive groups form Schurian equivalenced non-regular schemes (see [56]), which form a class of *pseudocyclic* association schemes. To date, there are not many known constructions of pseudocyclic schemes, and Theorem 1.2 implies that there are no new examples in this subclass.

In the next result, we call a transitive permutation group *strongly* $\frac{3}{2}$ -transitive if all non-principal constituents of the permutation character are distinct and have the same degree. Theorem 30.2 of [68] states that strongly $\frac{3}{2}$ -transitive groups are either abelian and regular, or $\frac{3}{2}$ -transitive. Hence Theorem 1.2, together with Lemma 6.2, implies the following.

Corollary 1.5. *The strongly $\frac{3}{2}$ -transitive almost simple permutation groups are precisely the groups in parts (i), (ii) and (iii) of Theorem 1.2.*

A related notion arose in a paper of Dixon [24]. He defines a $\mathbb{Q}I$ -group to be a finite transitive permutation group for which the permutation character is $1 + \theta$ where θ is irreducible over the rationals. All such groups are primitive and strongly $\frac{3}{2}$ -transitive. Moreover, Dixon reduced their study to the almost simple case. The groups in (iii) of Theorem 1.2 satisfy the $\mathbb{Q}I$ -condition if and only if $q-1 = 2^f - 1$ is prime (see [24, Theorem 11]). Hence we have

Corollary 1.6. *The almost simple $\mathbb{Q}I$ -groups which are not 2-transitive are precisely the groups in part (iii) of Theorem 1.2 with $2^f - 1$ prime.*

The final consequence concerns a connection between Theorem 1.3 and *triple factorizations*. These are factorizations of a group G as a product ABA for subgroups A, B ; such factorizations have been of interest since the paper of Higman and McLaughlin [29] linking them with incidence geometries.

It is an elementary observation that if G is a transitive permutation group with point stabiliser H , and p is a prime dividing $|H|$, then all the subdegrees of G are coprime to p if and only if G admits the

triple factorization

$$G = HN_G(P)H,$$

where P is a Sylow p -subgroup of H . Indeed, for $g \in G \setminus H$ the condition on the subdegrees implies that there is an H -conjugate P^h of P contained in the two-point stabiliser $H \cap H^g$. Then $P^h, P^{hg^{-1}}$ are contained in H , hence are H -conjugate, and so $hg^{-1}h' \in N_G(P)$ for some $h' \in H$, giving $g \in HN_G(P)H$. Hence the subdegree condition implies the triple factorization, and the converse implication is proved by reversing the argument.

In view of Theorem 1.3, this gives the following result concerning triple factorizations of groups of Lie type. We denote the set of Sylow p -subgroups of a group G by $\text{Syl}_p(G)$.

Corollary 1.7. *Let G be an almost simple group with socle L of Lie type in characteristic p , and let H be a maximal subgroup of G . Assume that p divides $|H|$. Then $G = HN_G(P)H$ for $P \in \text{Syl}_p(H)$ if and only if G, H are as in Theorem 1.3(ii)-(iv).*

We note that in [1, 2] it is shown that there are many more triple factorizations $G = HAH$ with A a maximal subgroup properly containing $N_G(P)$, where H is a maximal subgroup of order divisible by p and $P \in \text{Syl}_p(H)$.

2. BASIC LEMMAS

In this section, we provide some of the lemmas that we repeatedly use in the course of this work. Let G be a transitive permutation group on a set Ω . We refer to $|\Omega|$ as the *degree* of G and the orbits of a point stabiliser as *suborbits*. Recall from the Introduction the result of Wielandt that a finite $\frac{3}{2}$ -transitive permutation group is primitive or a Frobenius group.

Lemma 2.1. *Let G be a finite transitive permutation group with a point stabiliser H .*

- (i) *Suppose the degree of G is divisible by $r > 1$. If G has a subdegree divisible by r then G is not $\frac{3}{2}$ -transitive.*
- (ii) *Suppose that $H < K < G$ and in the action of K on the set of right cosets of H , K has a suborbit of length ℓ . Then G has a suborbit of length ℓ .*
- (iii) *Let T be a normal subgroup of G . Let $g \in G$ and suppose $|T \cap H : T \cap H \cap H^g|$ is divisible by a positive integer k . Then $|H : H \cap H^g|$ is divisible by k .*
- (iv) *Let p be a prime such that H has a nontrivial normal p -subgroup P . Then G has a subdegree divisible by p .*

Proof. (i) This is clear.

(ii) Since K has a suborbit of length ℓ , there exists $g \in K$ such that $|H : H \cap H^g| = \ell$. Since $g \in G$, it follows that G has a suborbit of length ℓ .

(iii) Consider the group action of G on the right cosets of H . Since $T \cap H$ is a normal subgroup of H , the orbit lengths of $T \cap H$ divide the orbit lengths of H . Therefore, for all $g \in G$, we have that $|T \cap H : T \cap H \cap H^g|$ divides $|H : H \cap H^g|$.

(iv) There must be some H -orbit Δ upon which P acts nontrivially. Then as $P \triangleleft H$, all orbits of P on Δ have the same size. It follows that p divides $|\Delta|$. \square

Let G be a transitive permutation group with point stabiliser H . We say that a subgroup H_0 of H is *weakly closed* in G if whenever $H_0^g \leq H$ for $g \in G$ there exists $h \in H$ such that $H_0^g = H_0^h$.

Lemma 2.2. *Let G be a transitive permutation group with point stabiliser H and let p be a prime. Suppose there exists $T \leq H$ such that*

- (i) $N_G(T) \not\leq H$, and
- (ii) *for all $S \in \text{Syl}_p(H)$, the group $\langle T, S \rangle$ contains a normal subgroup H_0 of H which is weakly closed in G such that $H = N_G(H_0)$.*

Then G has a subdegree which is divisible by p .

Proof. Let $g \in N_G(T) \setminus H$. Then $T \leq H \cap H^g$. Suppose that $|H : H^g \cap H|$ is coprime to p . Then there exists $S \in \text{Syl}_p(H)$ such that $S \leq H^g \cap H$. Thus $H^g \cap H \geq \langle S, T \rangle \geq H_0$ and so $H_0^{g^{-1}} \leq H$. Since H_0 is weakly closed in G , it follows that there exists $h \in H$ such that $H_0^{g^{-1}} = H_0^h$. Thus $hg \in N_G(H_0) = H$. Hence $g \in H$, a contradiction, and so G has a subdegree divisible by p . \square

The next result is commonly known as Tits' Lemma. A proof can be found in [64, (1.6)].

Lemma 2.3. *Let L be a quasisimple group of Lie type in characteristic p , and let H be a maximal subgroup of L which has index coprime to p and does not contain the unique quasisimple normal subgroup of L . Then H is a parabolic subgroup.*

Lemma 2.4. *Let G be a group of Lie type of characteristic p , with p odd. Let G act primitively with point stabiliser H . Suppose that there exists $H_0 \triangleleft H$, weakly closed in G and H_0 a central product of quasisimple Lie type groups of characteristic p , excluding types $\mathrm{PSL}_3(q)$, $\mathrm{PSL}_d(q)$ ($d \geq 5$) and $E_6(q)$. Assume $|G : H|$ is even. Then G has a subdegree divisible by p .*

Proof. Let T be a Sylow 2-subgroup of H . As $|G : H|$ is even, T is not a Sylow 2-subgroup of G and so $N_H(T) < N_G(T)$. Let H_0 be a central product of quasisimple Lie type groups H_i of characteristic p , with each H_i subject to the conditions in the lemma. Let S be a Sylow p -subgroup of H . Then $|H : \langle T, S \rangle|$ is coprime to p and hence, by Lemma 2.3, for each i , the projection of $M := H_0 \cap \langle T, S \rangle$ to H_i has image $\pi_i(M)$ equal to either H_i or a parabolic subgroup of H_i . By our assumption on H_i , each parabolic subgroup of H_i has even index [43]. Since T is a Sylow 2-subgroup of H and $H_0 \triangleleft H$, it follows that M contains a Sylow 2-subgroup of H_0 , and hence that $\pi_i(M) = H_i$ for each i . Then since M contains a Sylow p -subgroup $S \cap H_0$ of H_0 , we have $M = H_0$, so $H_0 \leq \langle T, S \rangle$. By the maximality of H , $N_G(H_0) = H$ and hence the result follows from Lemma 2.2. \square

Some care is required when applying Lemma 2.4 in the case where H_0 is a central product of classical groups, as factors isomorphic to $\mathrm{PSL}_d(q)$ may be hidden due to isomorphisms in the low dimensional cases. However, by [34, Prop. 2.9.1], if a classical group in characteristic p is isomorphic to $\mathrm{PSL}_d(p^f)^k$ for some d, p, f and k then $d = 2$ or 4 , and so Lemma 2.4 does apply.

Next we note the following result of Neumann and Praeger [57, Corollary 1].

Lemma 2.5. *Let $G \leq \mathrm{Sym}(\Omega)$. If there exists a k -set Γ of Ω such that there is no $g \in G$ with $\Gamma^g \cap \Gamma = \emptyset$ then Γ intersects nontrivially a G -orbit of length at most $k^2 - k + 1$.*

Lemma 2.6. *Let G be a permutation group with point stabiliser H , let S be a Sylow p -subgroup of H for some prime p , and let $x \in H$ have order a power of p .*

- (i) *If $|S^G| > |S^H|^2 - |S^H| + 1$ then G has a subdegree divisible by p .*
- (ii) *If $|x^G| > |x^G \cap H|^2 - |x^G \cap H| + 1$ then G has a subdegree divisible by p .*

Proof. First consider the action of G on S^G . If $|S^G| > |S^H|^2 - |S^H| + 1$, it follows from Lemma 2.5 applied to the action of G on $\mathrm{Syl}_p(G)$ by conjugation that there exists $g \in G$ such that $S^H \cap (S^H)^g = \emptyset$. Since S^H is the set of Sylow p -subgroups of H , it follows that $H \cap H^g$ does not contain a Sylow p -subgroup of H . Hence the subdegree $|H : H \cap H^g|$ is divisible by p .

Now assume $|x^G| > |x^G \cap H|^2 - |x^G \cap H| + 1$. By Sylow's Theorem, x is contained in some Sylow p -subgroup S of H , and hence each Sylow p -subgroup of H contains a conjugate of x . By Lemma 2.5 applied to the action of G on itself by conjugation, there exists $g \in G$ such that $(x^G \cap H) \cap (x^G \cap H)^g = \emptyset$. Hence $S^H \cap (S^H)^g = \emptyset$ and the argument of the previous paragraph applies. \square

We remark that [11, Lemma 9] is a weaker but useful version of Lemma 2.6.

3. THE REDUCTION TO AFFINE AND ALMOST SIMPLE GROUPS

In this section we prove Theorem 1.1, which states that a $\frac{3}{2}$ -transitive group is either almost simple or affine. First we consider primitive groups preserving a Cartesian decomposition.

The group $\mathrm{Sym}(\Delta) \mathrm{wr} S_k$ acts primitively in its natural product action on the set $\Omega = \Delta^k$ when $|\Delta| \geq 3$. Suppose that G is a primitive subgroup of $\mathrm{Sym}(\Delta) \mathrm{wr} S_k$ in this action. Then the image of G under the natural homomorphism to S_k is a transitive permutation group K on the set $\{1, \dots, k\}$. Let G_1 be the stabiliser in G of 1 in this action. Then $G_1 \leq \mathrm{Sym}(\Delta) \times (\mathrm{Sym}(\Delta) \mathrm{wr} S_{k-1})$ and so we have a natural homomorphism $\pi_1 : G_1 \rightarrow \mathrm{Sym}(\Delta)$ to the first direct factor. Let $H = \pi_1(G_1) \leq \mathrm{Sym}(\Delta)$. By [36, (2.2)], there exists $g \in \mathrm{Sym}(\Delta)^k \cap \ker(\pi_1)$ such that $G^g \leq H \mathrm{wr} K$ and so we may assume that $G \leq H \mathrm{wr} K$. Then G_1 induces H on the set of first entries of the points of Ω . Since G is primitive on Ω , H must be primitive on Δ and we refer to H as the *primitive component of G relative to the decomposition $\Omega = \Delta^k$* . Note that it may be possible to write $\Omega = \Delta^r$ for some $r \neq k$ and such a decomposition would give rise to a different primitive component.

Lemma 3.1. *Let G be a finite primitive permutation group such that $G \leq H \mathrm{wr} S_k$ acting in product action on $\Omega = \Delta^k$ with primitive component H on Δ and $k \geq 2$. Let $K \leq S_k$ be the image of the natural homomorphism $G \rightarrow S_k$.*

- (i) If H has a subdegree of length ℓ on Δ then G has a subdegree of length ℓk on Ω .
- (ii) If K has an orbit of length ℓ on the set of r -subsets of $\{1, \dots, k\}$ then G has a subdegree divisible by ℓ .

Proof. Let $\delta \in \Delta$ and let $\alpha = (\delta, \dots, \delta) \in \Omega$. Then $G_\alpha = G \cap (H_\delta \text{ wr } S_k)$. Let $\delta' \in \Delta \setminus \{\delta\}$ and let $\gamma = (\delta', \delta, \dots, \delta)$. Then $|\gamma^{G_\alpha}| = |(\delta')^{H_\delta}|k$ and so part (i) holds.

Let $J \subseteq \{1, \dots, k\}$ such that $|J^K| = \ell$. Let γ be the element of Ω such that for each $j \in J$ we have $(\gamma)_j = \delta'$ and for $j \notin J$, $(\gamma)_j = \delta$ (where $(\gamma)_j$ denotes the j^{th} coordinate). For $\gamma' \in \Omega$, let $\pi(\gamma') = \{i \mid (\gamma')_i \neq \delta\}$. Then $\{\pi(\gamma') \mid \gamma' \in \gamma^{G_\alpha}\} = J^K$. Moreover, the set of all $\gamma' \in \gamma^{G_\alpha}$ such that $\pi(\gamma') = J$ forms a block of imprimitivity for G_α on γ^{G_α} . Hence part (ii) follows. \square

If a primitive group G on a set Ω cannot be embedded into a wreath product in product action (with $k \geq 2$) then we call G a *basic primitive permutation group*. One interpretation of the O'Nan-Scott Theorem for primitive groups is that a basic primitive permutation group is either affine, almost simple or of diagonal type [16, Theorem 4.6]. By choosing an appropriate representation of Ω as a Cartesian power we may assume that the primitive component of a nonbasic primitive permutation group is one of the following three basic types:

- *Affine Type:* Here $\Omega = \text{GF}(p)^k$ and $G = N \rtimes G_0$ where N is the group of all translations and G_0 is an irreducible subgroup of $\text{GL}_k(p)$. If we also insist that G is basic then G_0 is a primitive linear group, that is, does not preserve a nontrivial direct sum decomposition of the vector space.
- *Almost Simple Type:* Here G is isomorphic to a subgroup of $\text{Aut}(T)$ containing $\text{Inn}(T)$ for some nonabelian simple group T .
- *Diagonal Type:* Here G has socle $N = T^k$ for some nonabelian simple group T and $k \geq 2$. Moreover, N_α is a full diagonal subgroup of N . Identify T with $\text{Inn}(T)$. Then Ω can be identified with T^{k-1} , and $T^k \leq G \leq A$ where

$$A = \{(a_1, \dots, a_k)\pi \mid \pi \in S_k, a_1 \in \text{Aut}(T), a_i \in \text{Inn}(T)a_1\}$$

and the action of A on T^{k-1} is given by

$$\begin{aligned} (t_2, \dots, t_k)^{(a_1, \dots, a_k)} &= (a_1^{-1}t_2a_2, \dots, a_1^{-1}t_ka_k) \\ (t_2, \dots, t_k)^{\pi^{-1}} &= (t_{1\pi}^{-1}t_{2\pi}, \dots, t_{1\pi}^{-1}t_{k\pi}) \end{aligned}$$

for all $(a_1, \dots, a_k)\pi \in A$ and $(t_2, \dots, t_k) \in \text{Inn}(T)^{k-1}$ where $t_1 = 1$. Also, either G acts primitively on the set of simple direct factors of N , or $k = 2$ and G fixes each simple direct factor setwise.

Before dealing with groups of diagonal type we need the following lemma.

Lemma 3.2. *Let T be a nonabelian simple group and let p be a prime dividing $|T|$. Then T has a conjugacy class of size divisible by p .*

Proof. Suppose that every conjugacy class in T has size coprime to p . Then every element of T is centralised by some Sylow p -subgroup of T and so given a Sylow p -subgroup S of T , $C_T(S)$ meets each conjugacy class nontrivially. Hence in the action of T on the set of cosets of $C_T(S)$ every element of T fixes some coset. Since every transitive group of degree at least 2 has a fixed point free element [16, p. 173], it follows that $C_T(S) = T$ and so $S \triangleleft T$. Since T is nonabelian simple it follows that $S = 1$, a contradiction. Hence T has a conjugacy class with size divisible by p . \square

Lemma 3.3. *Let G be a primitive permutation group of diagonal type and let p be a prime dividing $|\Omega|$. Then G has a subdegree divisible by p .*

Proof. Let $N = T^k$ be the socle of G . Then $|\Omega| = |T|^{k-1}$ and so p divides $|T|$. Moreover, $\text{Inn}(T) \triangleleft G_\alpha \leq \text{Aut}(T) \times S_k$. By Lemma 3.2, we can choose $t \in T$ such that $|t^T|$ is divisible by p . Then as $|(t, 1, \dots, 1)^{\text{Inn}(T)}| = |t^T|$ and $\text{Inn}(T) \triangleleft G_\alpha$ it follows that G has a subdegree divisible by p . \square

Theorem 3.4. *Let G be a primitive permutation group on a set Ω and let p be a prime such that p divides $|\Omega|$. Then one of the following holds:*

- (i) G has a subdegree divisible by p ,
- (ii) G is almost simple or affine,
- (iii) $G \leq H \text{ wr } K$ acting in product action on $\Omega = \Delta^k$ with primitive component H such that H is almost simple, H does not have a subdegree divisible by p and for each $r \leq k$, each orbit of K on r -subsets has size coprime to p .

Proof. Suppose that G does not have a subdegree divisible by p . If G is a basic primitive permutation group then by Lemma 3.3 and the preceding remarks it follows that G is either almost simple or affine. If $G \leq H \text{ wr } K$ in product action on $\Omega = \Delta^k$ then by our remarks above we may assume that H is affine, almost simple or of diagonal type. By Lemma 3.1, H does not have a subdegree divisible by p and for each $r \leq k$, each orbit of K on r -subsets has size coprime to p . Lemma 3.3 implies that H is not of diagonal type and so H is almost simple or affine. Since the wreath product of an affine group with a subgroup of S_k is an affine group the result follows. \square

We can now prove Theorem 1.1.

Proof of Theorem 1.1: Let G be a finite primitive $\frac{3}{2}$ -transitive group and let p be a prime dividing $|\Omega|$. Then G does not have a subdegree divisible by p and so by Theorem 3.4, G is either affine, almost simple or $G \leq H \text{ wr } S_k$ acting on Δ^k with H almost simple and $k \geq 2$. Suppose the latter holds. Let ℓ be a subdegree of H on Δ . Then since G is $\frac{3}{2}$ -transitive, Lemma 3.1(i) implies that all subdegrees of G have size ℓk and that all suborbits of H have length ℓ . Let $\alpha = (\delta, \dots, \delta) \in \Delta^k$ and let $\gamma = (\delta', \delta', \delta, \dots, \delta)$ for some $\delta' \neq \delta$. Since $|\gamma^{G_\alpha}| = \ell k$, [26, Theorem 1.1] implies that $(H, |\Delta|)$ is one of $(\text{PGL}_2(7), 21)$, $(\text{PGL}_2(9), 45)$, $(M_{10}, 45)$ or $(\text{P}\Gamma\text{L}_2(9), 45)$. None of these groups is $\frac{3}{2}$ -transitive, which implies that G is also not $\frac{3}{2}$ -transitive. \square

4. ALTERNATING GROUPS

In this section we prove

Theorem 4.1. *The only primitive $\frac{3}{2}$ -transitive permutation groups with socle A_n ($n \geq 5$) that are not 2-transitive are the groups A_7 and S_7 acting on the set of 2-subsets of a 7 element set.*

Let G be an almost simple group with socle A_n ($n \geq 5$). Then either G is A_n or S_n , or $n = 6$ when there are three further possibilities. We shall handle the latter at the end of the section, so suppose now that $G = A_n$ or S_n .

The following result on the maximal subgroups of G can be found in [16, §4.6].

Proposition 4.2. *The maximal subgroups of G are the intersections with G of the following:*

INTRANSITIVE: subgroups $S_k \times S_{n-k}$ with $1 \leq k < n/2$.

IMPRIMITIVE: subgroups $S_k \text{ wr } S_{n/k}$ in imprimitive action on $\{1, \dots, n\}$.

PRIMITIVE:

- (i) $S_k \text{ wr } S_r$ ($n = k^r$, $k \geq 5$, $r \geq 2$) in product action on $\{1, \dots, n\}$;
- (ii) $\text{AGL}_d(p)$ ($n = p^d$);
- (iii) Diagonal groups $T^k \cdot (\text{Out}(T) \times S_k)$, where T is nonabelian simple, $n = |T|^{k-1}$;
- (iv) Almost simple groups acting primitively on $\{1, \dots, n\}$.

We now embark on the proof of Theorem 4.1. We first treat the case where the point stabiliser in G is a subgroup of intransitive type.

Lemma 4.3. *Let $G = A_n$ or S_n , where $n \geq 5$, in its natural action on k -sets with $k < n/2$. Then G is $\frac{3}{2}$ -transitive if and only if either $k = 1$, or $n = 7$ and $k = 2$.*

Proof. Let Ω be the set of all k -sets of $\{1, \dots, n\}$ and let $\alpha = \{1, \dots, k\} \in \Omega$. If $k = 1$ then G is 2-transitive on Ω and hence $\frac{3}{2}$ -transitive. Now assume that $k \geq 2$. Then G_α has $k+1$ orbits on Ω . These orbits are the sets $\Omega_i = \{\beta \in \Omega : |\beta \cap \alpha| = i\}$ for $0 \leq i \leq k$. Since for each i , there are $\binom{k}{i}$ subsets of α of size i , and $\binom{n-k}{k-i}$ subsets in the complement of α of size $k-i$, it follows that

$$|\Omega_i| = \binom{k}{i} \binom{n-k}{k-i}$$

for each i .

Now $|\Omega_{k-2}|/|\Omega_{k-1}| = (n-k-1)(k-1)/4$. For G to be $\frac{3}{2}$ -transitive this fraction must be equal to 1 and so $(n-k-1)(k-1) = 4$. Thus $n-k-1 = 1, 2$ or 4 and $k-1 = 4, 2, 1$ respectively. If $k = 5$, then $n = 7$, contradicting $k < n/2$. Also, if $k = 3$ then $n = 6$, another contradiction. Thus $k = 2$ and $n = 7$. Moreover, since $k = 2$ it follows that Ω_0 and Ω_1 are the only orbits of G_α on $\Omega \setminus \{\alpha\}$ and so G is $\frac{3}{2}$ -transitive in this case. \square

Now we turn to the case where the point stabiliser is of imprimitive type.

Lemma 4.4. *Let $G = A_{k\ell}$ or $S_{k\ell}$, with $k, \ell \geq 2$ and $k\ell \geq 5$, act on the set of partitions of a $k\ell$ -set into ℓ parts of size k . Then G is $\frac{3}{2}$ -transitive if and only if (k, ℓ) is $(3, 2)$. In this situation, G is in fact 2-transitive.*

Proof. Let $\Omega = \{1, \dots, k\ell\}$ and $\mathcal{P}_1 = \{\Delta_1, \dots, \Delta_\ell\}$ be the partition of Ω with $\Delta_i = \{(i-1)k+1, \dots, ik\}$. Then

$$H := G_{\mathcal{P}_1} = ((\text{Sym}(\Delta_1) \times \text{Sym}(\Delta_2) \times \dots \times \text{Sym}(\Delta_\ell)) \rtimes S_\ell) \cap G \cong (S_k \text{ wr } S_\ell) \cap G.$$

Suppose first that $k \geq 3$ and let $\mathcal{P}_2 = \{\Lambda_i\}$ where for $i = 1, \dots, \ell-1$, $\Lambda_i = \{(i-1)k+2, \dots, ik+1\}$ and $\Lambda_\ell = \{1, (\ell-1)k+2, \dots, k\ell\}$. Then $H \cap G_{\mathcal{P}_2} \cong (S_{k-1} \text{ wr } C_\ell) \cap G$. Hence $|\mathcal{P}_2^H| = k^\ell(\ell-1)!$. Next let $\mathcal{P}_3 = \{\{3, 4, \dots, k+2\}, \{1, 2, k+3, \dots, 2k\}, \Delta_3, \dots, \Delta_\ell\}$. Thus $H \cap G_{\mathcal{P}_3}$ is isomorphic to

$$(J \times (S_k \wr S_{\ell-2})) \cap G,$$

where J is the stabiliser in $S_k \text{ wr } S_2$ of the partition

$$\{1, 2\}, \{k+1, k+2\}, \{3, \dots, k\}, \{k+3, \dots, 2k\}$$

of $\{1, 2, \dots, 2k\}$. If $k-2 = 2$, then $J \cong S_2 \wr D_8$, otherwise, $J \cong (S_2 \times S_{k-2}) \wr S_2$. Hence $|\mathcal{P}_3^H| = 9\ell(\ell-1)/2$ if $k = 4$ or $k^2(k-1)^2\ell(\ell-1)/8$ otherwise. One can check that for $\ell \geq 4$, this is less than $|\mathcal{P}_2^H|$ and so G is not $\frac{3}{2}$ -transitive in these cases. For $\ell = 2, 3$, equality only holds for $\ell = 2$ and $k = 3$. In this case G is 2-transitive.

Suppose now that $k = 2$ and note that $\ell \geq 3$. Let $\mathcal{P}_4 = \{\{1, 4\}, \{2, 3\}, \Delta_3, \dots, \Delta_\ell\}$. Then $H \cap G_{\mathcal{P}_4} = (D_8 \times S_2 \text{ wr } S_{\ell-2}) \cap G$ and so $|\mathcal{P}_4^H| = \ell(\ell-1)/2$. Now let $\mathcal{P}_5 = \{\{2, 3\}, \{4, 5\}, \{6, 1\}, \Delta_4, \dots, \Delta_\ell\}$. Then $H \cap G_{\mathcal{P}_5} = (S_3 \times (S_2 \text{ wr } S_{\ell-3})) \cap G$. Hence $|\mathcal{P}_5^H| = 4\ell(\ell-1)(\ell-2)/3$ and so G is not $\frac{3}{2}$ -transitive. \square

Given $g \in S_n$, we define the *support* of g as $\text{supp}(g) = \{i \in \{1, \dots, n\} \mid i^g \neq i\}$. The *minimal degree* of a permutation group H is the minimal size of the support of a nontrivial element of H .

Lemma 4.5. *Let $a, b \in S_n$. Then $|\text{supp}([a, b])| \leq 2|\text{supp}(a)|$.*

Proof. Let $i \in \{1, \dots, n\}$. If neither i nor $i^{b^{-1}}$ belong to $\text{supp}(a)$, then $i^{[a, b]} = i^{b^{-1}ab} = i^{b^{-1}b} = i$, that is, $[a, b]$ fixes i . The result follows. \square

Lemma 4.6. *Let $G = A_n$ or S_n and let H be a primitive subgroup of G not containing A_n .*

- (i) *If $n \geq 26$, then the minimal degree of H is at least 11.*
- (ii) *If $n < 26$, then the transitive action of G on the right cosets of H is either 2-transitive or not $\frac{3}{2}$ -transitive.*

Proof. Part (i) follows from classical results on primitive groups with small minimal degree (see [68, §15]). Part (ii) was verified using the primitive groups library in GAP [25]. \square

The remaining case in the proof of Theorem 4.1 is that in which a point stabiliser in G is primitive in the natural action of degree n .

Lemma 4.7. *Let $G = A_n$ or S_n and let H be a primitive subgroup of G such that H does not contain A_n and $|H|$ is even. If the transitive action of G on the set of right cosets of H is $\frac{3}{2}$ -transitive, then it is 2-transitive.*

Proof. By Lemma 4.6, we can assume that $n \geq 26$ and the minimal degree of H is at least 11. Let $x = (12345) \in A_n$. Since H is primitive, and the minimal degree is at least 11, we have $x \notin H$. Let $g \in H \cap H^x$. Then $g, g^{x^{-1}} \in H$. However, by Lemma 4.5, $[g, x] = 1$, that is, x centralises g . Hence $H \cap H^x = C_H(x)$.

Next let $y = (12)(34) \in A_n$ and let $g \in H \cap H^y$. Arguing as in the previous paragraph, $H \cap H^y = C_H(y)$. Now $|C_H(x)| = |H_{12345}| |C_H(x) : H_{12345}|$ and $|C_H(x) : H_{12345}|$ is 1 or 5. Furthermore, the size of $C_H(y)$ is $|C_H(y) : H_{1234}| |5^{H_{1234}}| |H_{12345}|$. Since $|H|$ is even, it contains an element z of order two with at least six 2-cycles (for the minimal degree is at least 11). Replacing H by a conjugate if necessary, we may assume that (12) and (34) are 2-cycles of z , so $z \in C_H(y)$. Hence $|C_H(y) : H_{1234}| \in \{2, 4, 8\}$. Thus $|C_H(y)| \neq |C_H(x)|$ and so $|H : H \cap H^x| \neq |H : H \cap H^y|$. \square

Lemma 4.8. *Let $p \geq 5$ be an odd prime and let $H = \text{AGL}_1(p) \cap A_p$. Then the action of A_p on the set of right cosets of H in A_p is $\frac{3}{2}$ -transitive if and only if $p = 5$ and in that case the action is 2-transitive.*

Proof. If $p = 5$ then the action is the 2-transitive action of A_5 of degree 6. If $p = 7$ a quick calculation shows that there are suborbits of lengths 7 and 21. Hence we may assume that $p \geq 11$. Let $x = (123) \in A_p$. Since H has no nontrivial elements of support at most 6 it follows from Lemma 4.5 that $H \cap H^x = C_H(x)$. Moreover, $|C_H(x)| = |C_H(x) : H_{1,2,3}| |H_{1,2,3}|$. Since no nontrivial element of H fixes more than one point it follows that $|C_H(x)| = 1$ or 3. Thus H has an orbit of length $p(p-1)/2$ or $p(p-1)/6$ on Ω . Let $g \in H$ have order $(p-1)/2$ such that h has two cycles of length $(p-1)/2$. Since $(p-1)/2 \geq 5$, there exists $y \in A_p$ which normalises, but does not centralise, $\langle g \rangle$. As H is self-normalising in A_p and $\langle g \rangle$ is maximal in H it follows that $H \cap H^y = \langle g \rangle$. Thus H has an orbit of length p on the set of right cosets of H and so A_p is not $\frac{3}{2}$ -transitive. \square

We note that an alternative approach to the case where H acts primitively on $\{1, \dots, n\}$ would be to use the result of [11] that the action has a regular suborbit when $n > 12$ and hence is not $\frac{3}{2}$ -transitive.

Proof of Theorem 4.1: Let $G = A_n$ or S_n be $\frac{3}{2}$ -transitive but not 2-transitive on the set of cosets of a maximal subgroup H . Then H is primitive, imprimitive or intransitive of degree n . If H is primitive, then Lemma 4.7 implies that $|H|$ is odd. By Proposition 4.2 this forces $n = p$, $G = A_p$ and $H = \text{AGL}_1(p) \cap A_p$ for some odd prime p . By Lemma 4.8, this action is not $\frac{3}{2}$ -transitive, unless $p = 5$ and in that case it is 2-transitive. Lemma 4.4 shows that H is not imprimitive. Hence H is intransitive, and Lemma 4.3 implies that G is A_7 or S_7 acting on 21 points.

It remains to handle the extra possibilities for G when $n = 6$. Apart from A_6 and S_6 , the groups with socle A_6 are M_{10} , $\text{PGL}_2(9)$ and $\text{Aut}(A_6)$. These are easily checked using the Atlas [22]. \square

5. CLASSICAL GROUPS: PRELIMINARIES

In this section and the next, we prove

Theorem 5.1. *Let G be a finite almost simple primitive permutation group on Ω with socle a classical simple group L of characteristic p . Let H be the stabiliser in G of a point in Ω .*

- (A) *If G has no subdegrees divisible by p , then either $|H|$ is not divisible by p or one of the following holds:*
- (i) $L = \text{PSL}_2(q)$, q is even, $H = N_G(D_{2(q+1)})$ and either $|G : L|$ is odd or $q = 4$.
 - (ii) $G = \text{Sp}_{2d}(2)$ and $H = O_{2d}^\pm(2)$ with $d \geq 3$.
 - (iii) L and H are given in Table 3.

TABLE 3.

L	$ \Omega $	H	Conditions	Subdegrees
$\text{Sp}_4(2)'$	6	$N_G(A_5)$		1, 5
$\text{Sp}_4(2)'$	10	$N_G(C_3^2)$		1, 9
$\text{PSU}_3(5)$	50	$N_G(A_7)$		1, 7, 42
$\text{PSL}_2(9)$	6	$N_G(A_5)$		1, 5
$\text{PSL}_4(2)$	8	$N_G(A_7)$		1, 7
$\text{PSp}_4(3)$	27	$N_G(2^4.A_5)$		1, 10, 16
$\text{PSL}_3(2)$	8	$C_7 \rtimes C_6$	$G = L.2$	1, 7

- (B) *If G is $\frac{3}{2}$ -transitive, and not 2-transitive, then (A)(i) holds with $q = 2^f \geq 8$ and either $G = L$ or $|G : L| = f$ is prime.*

In the proof of this theorem we shall need some preliminary information on subgroups and conjugacy classes in classical groups.

Proposition 5.2. *Let $G = \text{SL}_d(q)$, $\text{SU}_d(q)$, $\text{Sp}_d(q)$ or $\Omega_d^\epsilon(q)$, with $d \geq 3$ in the last three cases and $q = p^f$ for some prime p . Let V be the natural module for G . Then there exists a cyclic subgroup T of G with order given in Table 4 such that $\langle S, T \rangle = G$ for every Sylow p -subgroup S of G . Moreover, T acts irreducibly on an ℓ -dimensional subspace W of V with ℓ given in Table 4 and trivially on a complement of W . When $G = \text{SU}_d(q)$, $\text{Sp}_d(q)$ or $\Omega_d^\epsilon(q)$ we can take W to be nondegenerate.*

Proof. Suppose first that $G = \text{SL}_d(q)$, $\text{Sp}_d(q)$, $\Omega_d^-(q)$ or $\text{SU}_d(q)$, with the additional assumption that d is odd when $G = \text{SU}_d(q)$. Let T be the intersection of G and a Singer cycle of $\text{GL}(V)$. Then by [6, Table 1], $|T|$ is as given in Table 4 and T is irreducible. Let $S \in \text{Syl}_p(G)$. By Tits' Lemma 2.3, all overgroups

TABLE 4.

G	$ T $	ℓ
$\mathrm{SL}_d(q)$	$(q^d - 1)/(q - 1)$	d
$\mathrm{SU}_d(q)$, d odd	$(q^d + 1)/(q + 1)$	d
$\mathrm{SU}_d(q)$, $d \geq 4$ even	$(q^{d-1} + 1)/(q + 1)$	$d - 1$
$\mathrm{Sp}_d(q)$	$q^{d/2} + 1$	d
$\Omega_d(q)$, $d \geq 3$ odd	$(q^{(d-1)/2} + 1)/(2, q + 1)$	$d - 1$
$\Omega_d^-(q)$, $d \geq 4$	$(q^{d/2} + 1)/(2, q + 1)$	d
$\Omega_d^+(q)$, $d \geq 4$	$(q^{d/2-1} + 1)/(2, q + 1)$	$d - 2$

of S other than G are contained in parabolic subgroups and hence fix a subspace. So it follows that $\langle T, S \rangle = G$.

Suppose next that $G = \mathrm{SU}_d(q)$ with $d \geq 4$ even. Let U be a nondegenerate hyperplane in V and let x be a nonsingular vector such that $U = \langle x \rangle^\perp$. Since $\mathrm{SU}_{d-1}(q) \leq G_U$ and $d - 1$ is odd, it follows that G has a cyclic subgroup T of order $(q^{d-1} + 1)/(q + 1)$ which acts irreducibly on U and fixes x . Let $S \in \mathrm{Syl}_p(G)$. If $G \neq \langle S, T \rangle$ then $\langle S, T \rangle$ fixes a totally isotropic subspace W (by Tits Lemma 2.3). Since T acts irreducibly on U , we have $W \cap U = \{0\}$ and so $W = \langle u + \lambda x \rangle$ for some nonzero vector $u \in U$ and $\lambda \in \mathrm{GF}(q) \setminus \{0\}$. Since T fixes x , it follows that T fixes u , contradicting T acting irreducibly on U . Hence $G = \langle S, T \rangle$.

Next suppose that $G = \Omega_d(q)$ with d and q odd. Let U be a nondegenerate hyperplane upon which the restriction of the quadratic form is elliptic (i.e. of minus type) and let x be a nonsingular vector such that $U = \langle x \rangle^\perp$. Since $\Omega_{d-1}^-(q) \leq G_U$ it follows that G has a cyclic subgroup T of order $(q^{(d-1)/2} + 1)/(2, q + 1)$ which acts irreducibly on U and fixes x . Let $S \in \mathrm{Syl}_p(G)$. Then by the same argument in the previous paragraph, $\langle T, S \rangle = G$.

Finally, suppose that $G = \Omega_d^+(q)$, with $d \geq 4$. Let U be a nondegenerate subspace of codimension 2 upon which the restriction of the quadratic form is elliptic. Then U^\perp does not contain any singular nonzero vectors. Since $\Omega_{d-2}^-(q) \leq G_U$ we have that G contains a cyclic subgroup T of order $(q^{d/2-1} + 1)/(2, q + 1)$ which acts irreducibly on U and trivially on U^\perp . Let $S \in \mathrm{Syl}_p(G)$. If $G \neq \langle S, T \rangle$ then $\langle S, T \rangle$ fixes a totally singular subspace W (again by Tits' Lemma 2.3). Since T acts irreducibly on U , it follows that $W \cap U = \{0\}$. Also $W \cap U^\perp = \{0\}$ as U^\perp does not contain any nonzero singular vectors. Hence either $W = \langle v + x \rangle$ or $\langle v + x, w + y \rangle$ where v and w are linearly independent vectors of U and $\langle x, y \rangle = U^\perp$. Since T fixes x , it follows that T fixes v , contradicting T being irreducible on U . Hence $G = \langle S, T \rangle$. \square

Lemma 5.3. *Let G be a classical group with normal subgroup X as in Proposition 5.2 such that $X \neq \mathrm{SL}_d(q)$, and let U be a proper nondegenerate subspace of the natural G -module V . Then p divides $|G : G_U|$.*

Proof. Since $G \cap \mathrm{GL}(V)$ is transitive on U^G , we may assume that $G \leq \mathrm{GL}(V)$. As U is nondegenerate, G fixes the decomposition of V given by $U \perp U^\perp$ with U^\perp also nondegenerate. By [34, Lemma 4.1.1], the groups induced by G_U on U and U^\perp contain Sylow p -subgroups of the isometry groups of these spaces. The p -parts of $|G_U^U|$, $|G_U^{U^\perp}|$ and $|G|$ can be read off from [34, Table 2.1C] and it is easily computed that $|G_U^U|_p |G_U^{U^\perp}|_p < |G|_p$. \square

Lemma 5.4. *Let G and V be as in Lemma 5.3 of unitary, symplectic or orthogonal type, and let U be a nondegenerate proper subspace of V of dimension $m \geq 2$. Suppose also that*

- (1) *if G is symplectic then $m \geq 4$,*
- (2) *if G is orthogonal then $m \geq 3$ and $d - m \geq 2$.*

Then there exists $W \in U^G$ such that $0 \neq W \cap U < U$ and $W \cap U$ is nondegenerate. Moreover, p divides $|G_U : G_{U,W}|$.

Proof. Suppose first that G is not symplectic and, if m, q are both even, suppose also that G is not orthogonal. Then there exists a nondegenerate proper subspace Y of U with codimension 1. By [34, Lemma 4.1.1 and Proposition 2.10.6], G_Y is irreducible on Y^\perp . (Note that if G is orthogonal then $\dim(Y^\perp) \geq 3$.) Thus G_Y is not contained in G_U and so there exists $g \in G_Y$ such that $Y \leq U^g \neq U$. Hence $Y = U \cap U^g$. Let $W = U^g$. By [34, Lemma 4.1.1], G_U induces a nontrivial classical group on U and hence by Lemma 5.3, $|G_U : G_{U,W}|$ is divisible by p . (Note that if G is orthogonal then $m \geq 3$.)

Suppose now that G is symplectic, or G is orthogonal with m and q both even. By hypothesis this implies $m \geq 4$. Let $Y_1 = \langle e_1, f_1 \rangle \leq U$ be a hyperbolic pair and let $Y_2 = \langle e_d, f_d \rangle$ be a hyperbolic pair in U^\perp . Then by Witt's Lemma and [34, Lemma 4.1.1], there exists $g \in G$ interchanging Y_1 and Y_2 while

fixing setwise $Y_1^\perp \cap U$. Then letting $W = U^g$ we have $U \cap W = Y_1^\perp \cap U$, a nondegenerate subspace of V . Hence by Lemma 5.3, $|G_U : G_{U,W}|$ is divisible by p . \square

Next we present a useful lemma concerning the subdegrees of groups of Lie type in parabolic actions.

Lemma 5.5. *Let G be an almost simple group with socle L of Lie type in characteristic p . Let P be a maximal parabolic subgroup of G . Exclude the following cases:*

$$\begin{aligned} L &= \mathrm{PSL}_d(q) \text{ with } G \leq \mathrm{P}\Gamma\mathrm{L}_d(q) \\ L &= \mathrm{P}\Omega_{2m}^+(q), m \text{ odd}, P = P_{m-1} \text{ or } P_m \\ L &= E_6(q), P = P_i \quad (i = 1, 3, 5, 6). \end{aligned}$$

Then in its action on the cosets of P , the group G has a unique nontrivial subdegree that is a power of p .

Proof. This follows from [45, 3.9]: except in the excluded cases, the parabolic P^- opposite to P is G -conjugate to P and the required suborbit is the P -orbit containing P^- . \square

Next we present some information on conjugacy class sizes in classical groups.

Definition 5.6. *Let $x \in \mathrm{GL}_d(q)$ and V be a d -dimensional vector space over $\mathrm{GF}(q)$. Let K be the algebraic closure of $\mathrm{GF}(q)$ and $\overline{V} = V \otimes K$, a d -dimensional vector space over K . Then x acts naturally on \overline{V} and we define $\nu(x)$ to be the codimension of the largest eigenspace of x on \overline{V} . For $x \in \mathrm{PGL}_d(q)$, we define $\nu(x)$ to be $\nu(\hat{x})$, where \hat{x} is a preimage of x in $\mathrm{GL}_d(q)$.*

We denote the lower triangular matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ by J_2 and we use J_2^s to denote the block diagonal matrix with s copies of J_2 .

When q is even and G is a symplectic or orthogonal group the conjugacy classes of involutions are described in [4]. When $G = \mathrm{Sp}_d(q)$, for each odd positive integer $s \leq d/2$, there is one class of involutions with $\nu(x) = s$, denoted by b_s , while for each even positive integer $s \leq d/2$ there are two classes of involutions with $\nu(x) = s$, denoted a_s and c_s . By [4, (8.10)], the group $\mathrm{SO}_d^\epsilon(q)$ meets each of the $\mathrm{Sp}_d(q)$ -conjugacy classes of involutions except $\mathrm{SO}_d^-(q)$ contains no involutions of type $a_{d/2}$ for $d \equiv 0 \pmod{4}$. Only involutions of type a_s or c_s lie in $\Omega_d^\epsilon(q)$ and the involutions of type b_s lie in $\mathrm{SO}_d^\epsilon(q) \setminus \Omega_d^\epsilon(q)$. Moreover, two involutions of $\mathrm{SO}_d^\epsilon(q)$ that are conjugate in $\mathrm{Sp}_d(q)$ are conjugate under an element of $\Omega_d^\epsilon(q)$, except the $a_{d/2}$ -class in $\mathrm{SO}_d^+(q)$ which splits into two $\Omega_d^+(q)$ -classes denoted $a_{d/2}$ and $a'_{d/2}$ that are fused by $\mathrm{SO}_d^+(q)$.

Combining [9, Lemma 3.20 and Proposition 3.22] we obtain the following bounds on the lengths of conjugacy classes. We denote $\mathrm{PSL}_n(q)$ by $\mathrm{PSL}_n^+(q)$ and $\mathrm{PSU}_n(q)$ by $\mathrm{PSL}_n^-(q)$.

Proposition 5.7. *Let $G = \mathrm{PSL}_d^\epsilon(q), \mathrm{PSp}_d(q)$ or $\mathrm{P}\Omega_d^\epsilon(q)$ with $q = p^f$ for some prime p . Let $x \in G$ have order p and let $s = \nu(x)$. Then $|x^G| > f_i(d, s, q)$ with $i = 1 + \delta_{2,p}$ (where $\delta_{2,p}$ is the Kronecker delta function) and $f_i(d, s, q)$ as given in Tables 5 and 6.*

TABLE 5. Bounds on unipotent conjugacy classes for p odd

G	$f_1(d, s, q)$
$\mathrm{PSL}_d^\epsilon(q)$	$\frac{q}{2(q-\epsilon)(q+1)} \max\{q^{2s(d-s)}, q^{ds}\}$
$\mathrm{PSp}_d(q)$	$\frac{q}{4(q+1)} \max\{q^{s(d-s)}, q^{ds/2}\}$
$\mathrm{P}\Omega_d^\pm(q)$	$\frac{q}{8(q+1)} \max\{q^{s(d-s-1)}, q^{d(s-1)/2}\}$
$\mathrm{P}\Omega_d(q)$	$\frac{1}{4} \max\{q^{s(d-s-1)}, q^{d(s-1)/2}\}$

An outer automorphism of a finite simple group of Lie type can be written as the product of an inner automorphism, a diagonal automorphism, a field automorphism and a graph automorphism. We follow the conventions of [27, Definition 2.5.13] as to the definition of a field, graph or graph-field automorphism. In Table 7, these automorphisms are referred to being of type f , g and gf respectively.

Lemma 5.8. *Let $L = \mathrm{PSL}_d^\epsilon(q), \mathrm{PSp}_d(q)$ or $\mathrm{P}\Omega_d^\epsilon(q)$ where $q = p^f$ for some prime p and let $x \in \mathrm{Aut}(L) \setminus \mathrm{PGL}(V)$ of prime order r . Then $|x^L| > h(d, r, q)$ where $h(d, r, q)$ is as given by Table 7.*

Proof. This is from [9, Lemma 3.48]. \square

Lemma 5.9. *Let L be one of the groups in the first column of Table 8. Then the number of transvections in L is given by the second column of Table 8.*

TABLE 6. Bounds on unipotent conjugacy classes for p even

G	Conditions	x	$f_2(d, s, q)$
$\text{PSL}_d^\epsilon(q)$		$[J_2^s, I_{d-2s}]$	$\frac{q}{2(q-\epsilon)(q+1)} q^{2s(d-s)}$
$\text{PSp}_d(q)$		a_s	$\frac{1}{2} q^{s(d-s)}$
		b_s, c_s	$\frac{1}{2} q^{s(d-s+1)}$
$\text{P}\Omega_d^\pm(q)$	$(s, \epsilon) \neq (d/2, +)$	a_s	$\frac{1}{4} q^{s(d-s-1)}$
		c_s	$\frac{1}{4} q^{s(d-s)}$
$\text{P}\Omega_d^+(q)$	$s = d/2$	$a_{d/2}, a'_{d/2}$	$\frac{1}{4} q^{d(d-2)/4}$
		$c_{d/2}$	$\frac{1}{4} q^{d^2/4}$

TABLE 7. Bounds on conjugacy classes of outer automorphisms

L	Type	Conditions	$h(d, r, q)$
$\text{PSL}_d^\epsilon(q)$	f	$q = q_0^r, r > 2$ if $\epsilon = -$	$\frac{1}{2} \left(\frac{q}{q+1} \right)^{(1-\epsilon)/2} q^{(d^2-1)(1-\frac{1}{r})-1}$
	g	$r = 2, d$ odd	$\frac{1}{2} \left(\frac{q}{q+1} \right)^{(1-\epsilon)/2} q^{\frac{1}{2}(d^2+d-4)}$
	g	$r = 2, d > 2$ even	$\frac{1}{2} \left(\frac{q}{q+1} \right)^{(1-\epsilon)/2} q^{\frac{1}{2}(d^2-d-4)}$
	gf	$(r, q, \epsilon) = (2, q_0^2, +), d > 2$	$\frac{1}{2} q^{\frac{1}{2}(d^2-3)}$
$\text{PSp}_d(q)$	f	$q = q_0^r$	$\frac{1}{4} q^{d(d+1)(1-\frac{1}{r})/2}$
	gf	$(d, r, p) = (4, 2, 2), f$ odd	q^5
$\text{P}\Omega_d^\epsilon(q), d$ even	f	$q = q_0^r$	$\frac{1}{4} q^{d(d-1)(1-\frac{1}{r})/2}$
	gf	$(r, q, \epsilon) = (2, q_0^2, +)$	$\frac{1}{4} q^{d(d-1)/4}$
	gf	$(d, r, q, \epsilon) = (8, 3, q_0^3, +)$	$\frac{1}{4} q^{56/3}$
	g	$(d, r, \epsilon) = (8, 3, +)$	$\frac{1}{8} q^{14}$
$\text{P}\Omega_d(q), dq$ odd	f	$q = q_0^r$	$\frac{1}{4} q^{d(d-1)(1-\frac{1}{r})/2}$

TABLE 8. Transvections

L	Number of transvections
$\text{PSL}_d(q)$	$(q^d - 1)(q^{d-1} - 1)/(q - 1)$
$\text{PSU}_d(q)$	$(q^d - (-1)^d)(q^{d-1} - (-1)^{d-1})/(q + 1)$
$\text{PSp}_d(q)$	$q^d - 1$

6. CLASSICAL GROUPS: PROOF OF THEOREM 5.1

In this section we prove Theorem 5.1. Throughout, G is an almost simple group with socle $L = \text{PSL}_d(q), \text{PSU}_d(q), \text{PSp}_d(q)$ or $\text{P}\Omega_d^\epsilon(q)$ acting primitively on a set Ω with $H = G_\alpha$ for some $\alpha \in \Omega$. If $L = \text{PSU}_d(q)$ we assume that $d \geq 3$ and $(d, q) \neq (3, 2)$. For $L = \text{PSp}_d(q)$ we have $d \geq 4$ and $(d, q) \neq (4, 2)$. Finally, if L is an orthogonal group we assume that $d \geq 7$.

We denote the natural module of G by V and we let $\{v_1, \dots, v_d\}$ be a basis for V over $\text{GF}(q)$ (over $\text{GF}(q^2)$ when $L = \text{PSU}_d(q)$). For classical groups with socle other than $\text{PSL}_d(q)$ it is often convenient to use bases specific to the sesquilinear form B and/or quadratic form Q , preserved by the group as follows:

- When $L = \text{PSp}_d(q)$ we call a basis $\{e_1, \dots, e_{d/2}, f_1, \dots, f_{d/2}\}$ such that $B(e_i, e_i) = B(f_i, f_i) = 0$ for all i and $B(e_i, f_j) = \delta_{i,j}$ for all i, j , a *symplectic basis*.
- When $L = \text{PSU}_d(q)$ with d even we call a basis $\{e_1, \dots, e_{d/2}, f_1, \dots, f_{d/2}\}$ such that $B(e_i, e_i) = B(f_i, f_i) = 0$ for all i and $B(e_i, f_j) = \delta_{i,j}$ for all i, j , a *unitary basis*.
- When $L = \text{PSU}_d(q)$ with d odd, we call a basis $\{e_1, \dots, e_{(d-1)/2}, f_1, \dots, f_{(d-1)/2}, x\}$ such that $B(e_i, e_i) = B(f_i, f_i) = B(e_i, x) = B(f_i, x) = 0$ for all i , $B(e_i, f_j) = \delta_{i,j}$ and $B(x, x) = 1$ a *unitary basis*.
- When $L = \text{P}\Omega_d^+(q)$, we call a basis $\{e_1, \dots, e_{d/2}, f_1, \dots, f_{d/2}\}$ where $Q(e_i) = Q(f_i) = 0$ and $B(e_i, f_j) = \delta_{i,j}$ for all i, j a *hyperbolic basis*.
- When $L = \text{P}\Omega_d^-(q)$, we call a basis $\{e_1, \dots, e_{d/2-1}, f_1, \dots, f_{d/2-1}, x, y\}$ where $Q(e_i) = Q(f_i) = 0$, $B(e_i, f_j) = \delta_{i,j}$, $B(e_i, x) = B(e_i, y) = B(f_i, x) = B(f_i, y) = 0$ for all i, j , $Q(x) = 1$, $B(x, y) = 1$ and $Q(y) = \zeta$ where $\mathbf{x}^2 + \mathbf{x} + \zeta$ is irreducible over $\text{GF}(q)$, an *elliptic basis*.

- When $L = \text{P}\Omega_d(q)$ with d odd, we call a basis $\{e_1, \dots, e_{(d-1)/2}, f_1, \dots, f_{(d-1)/2}, x\}$ where $Q(e_i) = Q(f_i) = 0$, $B(e_i, f_j) = \delta_{i,j}$, $B(e_i, x) = B(e_i, y) = B(f_i, x) = 0$ for all i, j and $Q(x) \neq 0$, a *parabolic basis*.

The maximal subgroups of the classical groups are described by Aschbacher's Theorem [3]. They fall into eight families \mathcal{C}_i ($1 \leq i \leq 8$) of “geometric” subgroups, together with a further class \mathcal{C}_9 consisting of almost simple groups in absolutely irreducible representations satisfying certain extra conditions (see [34, p3]). We shall deal with each case $H \in \mathcal{C}_i$ in a separate subsection below. There are three further cases where extra possibilities for H have to be considered – those in which G contains a graph automorphism of $L = \text{PSL}_d(q)$ or $\text{Sp}_4(q)$ (q even), or a triality automorphism of $L = \text{P}\Omega_8^+(q)$. These are considered at the end of the section.

Throughout, we shall use the detailed descriptions of the subgroups in the families \mathcal{C}_i which can be found in [34, Chapter 4].

6.1. Aschbacher class \mathcal{C}_1 : Suppose $H \in \mathcal{C}_1$. Here H is the stabiliser of some subspace U of dimension m with $1 \leq m \leq d/2$. If $L = \text{PSL}_d(q)$ or U is totally singular, then H has a nontrivial normal p -subgroup and so by Lemma 2.1(iv), G has a p -subdegree (i.e. a subdegree divisible by p). If G is an orthogonal group and U is a nonsingular 1-space, then the subdegrees of G are given in [5] or [63, pp.331,332] from which we see that there is always a p -subdegree. This leaves us to deal with the case where $L \neq \text{PSL}_d(q)$ and U is a nondegenerate subspace of dimension m (with $m \geq 2$ if $L \neq \text{PSU}_d(q)$). Then H also stabilises U^\perp which has dimension $d - m$. Since G acts primitively on Ω , it follows that U and U^\perp are not similar. Hence if G is not orthogonal then $d - m > m$, while if G is orthogonal $d - m \geq m$, with equality implying that m is even with the restriction of the quadratic form to U being hyperbolic, while the restriction of the quadratic form to U^\perp is elliptic. Note that if G is symplectic, then both d and m are even and so $d - m \geq 4$ in this case, while if G is orthogonal then $d - m \geq 4$ and $m \geq 2$. Thus by Lemma 5.4, we can find $W \in (U^\perp)^G$ such that $0 \neq W \cap U^\perp < U^\perp$ and $W \cap U^\perp$ is nondegenerate. Moreover, p divides $|G_{U^\perp} : G_{U^\perp, W}|$. Since $W^\perp \in U^G$, $H_W = H_{W^\perp}$ and $G_U = G_{U^\perp}$ it follows that G has a p -subdegree. This proves (A) of Theorem 5.1 in the \mathcal{C}_1 case.

For $\frac{3}{2}$ -transitivity, if $L \neq \text{PSL}_d(q)$ and U is not totally singular, then p divides $|\Omega|$ and so the existence of a p -subdegree implies that G is not $\frac{3}{2}$ -transitive. If $L \neq \text{PSL}_d(q)$ and U is totally singular, then by Lemma 5.5, except in the case $L \neq \text{P}\Omega_d^+(q)$ with $d \equiv 2 \pmod{4}$ and $\dim U = d/2$, G has a unique subdegree which is a power of p . Hence if G is not 2-transitive then G is not $\frac{3}{2}$ -transitive.

For $L = \text{P}\Omega_d^+(q)$ with $m = d/2 \geq 5$ odd, let $\{e_1, \dots, e_{d/2}, f_1, \dots, f_{d/2}\}$ be a hyperbolic basis for V . Without loss of generality we may suppose that $U = \langle e_1, \dots, e_{d/2} \rangle$ and note that a maximal totally singular subspace W of V is in U^\perp if and only if $W \cap U$ has even codimension in U . Moreover, as G is primitive, $U^G = U^\perp$. Thus by [8, Lemma 9.4.2], the subdegrees are

$$q^{(m-i)(m-i-2)/2} \begin{bmatrix} m \\ i \end{bmatrix}_q,$$

for each i with $0 \leq i < m$ and $m - i$ even.¹ Hence G is not $\frac{3}{2}$ -transitive.

If $L = \text{PSL}_d(q)$ the suborbits are $\Omega_i = \{W \in \Omega \mid \dim(U \cap W) = i\}$ for each i , $0 \leq i \leq m$. By [8, Lemma 9.3.2(ii)],

$$|\Omega_0| = q^{m^2} \begin{bmatrix} d-m \\ m \end{bmatrix}_q.$$

When $m = 1$, G is 2-transitive. For $m \geq 2$,

$$\begin{aligned} |\Omega_{m-1}| &= q \begin{bmatrix} d-m \\ 1 \end{bmatrix}_q \begin{bmatrix} m \\ m-1 \end{bmatrix}_q \\ &= \frac{(q^m - 1)q(q^{d-m} - 1)}{(q - 1)(q - 1)} \end{aligned}$$

is another subdegree. Since $|\Omega_0|$ and $|\Omega_{m-1}|$ are not equal, it follows that G has two different subdegrees and so G is not $\frac{3}{2}$ -transitive.

6.2. Aschbacher class \mathcal{C}_2 : Here H is the stabiliser of a decomposition $V = U_1 \oplus \dots \oplus U_t$, where all $\dim U_i = m$, $d = mt$ and $t > 1$. Roughly speaking H is the wreath product of a classical group on U_1 with S_t . Detailed descriptions of the subgroups in \mathcal{C}_2 can be found in [34, §4.2]. There are several cases to consider.

¹By $\begin{bmatrix} m \\ i \end{bmatrix}_q$ we mean the Gaussian coefficient used to denote the number of i -subspaces in an m -dimensional vector space over $\text{GF}(q)$. We have

$$\begin{bmatrix} m \\ i \end{bmatrix}_q = \frac{(q^m - 1)(q^{m-1} - 1) \dots (q^{m-i+1} - 1)}{(q^i - 1) \dots (q - 1)}$$

6.2.1. $L = \text{PSL}_d(q)$. If $t > 2$, [33, Theorem 1.4] implies that there exists $g \in G$ such that $H \cap H^g \cap L = 1$. If $|H \cap L|$ is divisible by p (which will happen if $m > 1$) then it follows that G has a subdegree divisible by p . Moreover, as p divides $|\Omega|$, it follows that in these cases G is not $\frac{3}{2}$ -transitive. Thus we may assume, if $t > 2$ that p does not divide $|H \cap L|$ and $m = 1$.

Suppose next that $m = 1$ and that p divides $|H|$. Then either p divides $|H \cap L|$ (and hence $t = 2$), or H contains a field automorphism of order p . If p divides $|H \cap L|$ and $t = 2$, then we must have $p = 2$. In this case, let $x \in H \cap L \cong D_{2(q-1)}$ be an involution. Then $|x^G| = q^2 - 1$ while $|x^G \cap H| = q - 1$. Thus by Lemma 2.6(ii), G has an even subdegree. Suppose now that p does not divide $|H \cap L|$ and H contains a field automorphism x of order p . Note that p must be odd and $d < p$. Then Lemma 5.8 implies that $|x^G| > \frac{1}{2}q^{(2d^2-5)/3}$. Since $|H \cap L|$ is coprime to p , the Sylow p -subgroup of H is cyclic and so $\langle x \rangle^G \cap H = \langle x \rangle^H$. When $G = \text{Aut}(L)$ we have $|H| = (q-1)^{d-1}d!f2$ and $|C_H(x)| = (q^{1/p}-1)^{d-1}d!f2$, where $q = p^f$. Thus for all possibilities for G we have $|x^G \cap H| \leq (p-1)|x^H| \leq (p-1)(q-1)^{d-1}/(q^{1/p}-1)^{d-1}$. Hence for $d \geq 3$ we have $|x^G \cap H| \leq \frac{1}{2}q^{d-1}$ and Lemma 2.6(ii) implies that G has a subdegree divisible by p . For $d = 2$, we have $|x^G \cap H| < q$. Moreover, $C_L(x) = \text{PSL}_2(q^{1/p})$ and so in fact $|x^G| > q^{3-3/p} \geq q^2$. Thus Lemma 2.6(ii) again yields a p -subdegree. Since $|\Omega|$ is divisible by p , it follows that if p divides $|H|$ then G is not $\frac{3}{2}$ -transitive.

Now suppose $m = 1$ and p does not divide $|H|$. Then $d < p$ and p is odd. Let $V = \langle v_1 \rangle \oplus \dots \oplus \langle v_d \rangle$ be the decomposition stabilized by H , let γ be the decomposition $\langle v_1 + v_2 \rangle \oplus \langle v_2 \rangle \oplus \dots \oplus \langle v_d \rangle$ and δ be the decomposition $\langle v_1 + v_2 \rangle \oplus \langle v_1 - v_2 \rangle \oplus \langle v_3 \rangle \oplus \dots \oplus \langle v_d \rangle$. Then $|\gamma^H| = (q-1)d(d-1)$ and $|\delta^H| = (q-1)d(d-1)/2$, and so G is not $\frac{3}{2}$ -transitive.

Finally suppose that $t = 2$ and $m > 1$. Let $V = \langle v_1, \dots, v_m \rangle \oplus \langle v_{m+1}, \dots, v_{2m} \rangle$ be the decomposition fixed by H and let γ be the decomposition $\langle v_1, \dots, v_{m-1}, v_{m+1} \rangle \oplus \langle v_m, v_{m+2}, \dots, v_{2m} \rangle$. Then H_γ fixes the decomposition $\langle v_1, \dots, v_{m-1} \rangle \oplus \langle v_m \rangle \oplus \langle v_{m+1} \rangle \oplus \langle v_{m+2}, \dots, v_{2m} \rangle$. Hence p divides $|H : H_\gamma|$ (as H_{U_1} has a composition factor isomorphic to $\text{SL}_m(q)$ and $(H_{U_1})_\gamma$ stabilises a decomposition of U_1). Since p divides $|\Omega|$, G is not $\frac{3}{2}$ -transitive.

From now on, we suppose that L is not $\text{PSL}_d(q)$.

6.2.2. $t = 2$ and U_1, U_2 are maximal totally singular subspaces: Without loss of generality we may suppose that $U_1 = \langle e_1, \dots, e_m \rangle$ and $U_2 = \langle f_1, \dots, f_m \rangle$ with $m = d/2$ such that $B(e_i, f_j) = \delta_{ij}$. Suppose first that $m \geq 3$ and let $W_1 = \langle f_1, f_2, e_3, \dots, e_m \rangle$. Then $U_1 \cap W_1$ has codimension 2 in U_1 and lies in the same G -orbit as U_1 . (The only time G may not be transitive on the set of maximal totally isotropic subspaces is when G is an orthogonal group of plus type. In this case, two maximal totally isotropic subspaces lie in the same orbit if and only if they intersect in an even codimension subspace.) Let $W_2 = \langle e_1, e_2, f_3, \dots, f_m \rangle$. Then $\gamma = \{W_1, W_2\} \in \Omega$. It follows that H_{U_1, W_1} fixes $U_1 \cap W_1 = \langle e_3, \dots, e_m \rangle$ and $U_2 \cap W_1 = \langle f_1, f_2 \rangle$. Hence H_γ fixes the set $\{\langle e_1, e_2 \rangle, \langle e_3, \dots, e_m \rangle, \langle f_1, f_2 \rangle, \langle f_3, \dots, f_m \rangle\}$ and so $|H : H_\gamma|$ is divisible by p . When $m = 2$, G is not orthogonal and so G is transitive on the set of m -dimensional totally isotropic subspaces. Thus let $W_1 = \langle e_1, f_2 \rangle \in U_1^G$. Also let $W_2 = \langle e_2, f_1 \rangle$ and $\gamma = \{W_1, W_2\} \in \Omega$. Then H_γ fixes $\{\langle e_1 \rangle, \langle e_2 \rangle, \langle f_1 \rangle, \langle f_2 \rangle\}$ and so p divides $|H : H_\gamma|$. Thus we have found a p -subdegree in all cases and hence G is also not $\frac{3}{2}$ -transitive as $|\Omega|$ is divisible by p .

6.2.3. G is orthogonal, $t = 2$ and U_1 and U_2 are nondegenerate, similar but nonisometric subspaces: Here both q and $d/2$ are odd. By Lemma 5.4, there exists $W_1 \in U_1^G$ such that $\{0\} \neq U_1 \cap W_1 \neq W_1$ and $U_1 \cap W_1$ is nondegenerate. Let W_2 be a nondegenerate subspace complementary to W_1 , and isometric to U_2 , and let γ be the decomposition $V = W_1 \oplus W_2$ in Ω . Then $H_{U_1, \gamma}$ fixes the nondegenerate subspace $U_1 \cap W_1$ of U_1 and so p divides $|H : H_\gamma|$ (by Lemma 5.3). Since H is not parabolic, p divides $|\Omega|$ and so G is not $\frac{3}{2}$ -transitive.

The remaining cases to consider are where $V = U_1 \perp U_2 \perp \dots \perp U_t$ where the U_i are isometric nondegenerate subspaces of dimension m . We break up the analysis as follows.

6.2.4. $m \geq 3$, or $m = 2$ and G is unitary: Here Lemma 5.4 implies that there exists a nondegenerate subspace W_1 of $U_1 \perp U_2$ in U_1^G such that $W_1 \cap U_1$ is a proper nondegenerate subspace of U_1 . In this case, let $W_2 = W_1^\perp \cap (U_1 \perp U_2)$ and γ be the decomposition $V = W_1 \perp W_2 \perp U_3 \perp \dots \perp U_t$. Then $H_{U_1, U_2, \gamma}$ fixes $U_1 \cap W_1$. Since H_{U_1, U_2} induces a classical group on U_1 and $U_1 \cap W_1$ is nondegenerate, it follows by Lemma 5.3 that p divides $|H_{U_1, U_2} : H_{U_1, U_2, \gamma}|$ and hence also divides $|H : H_\gamma|$. Since p divides $|\Omega|$, G is not $\frac{3}{2}$ -transitive.

6.2.5. $m = 2$ and G is symplectic: Note that [34, Proposition 6.2.6] implies $q \neq 2$. Suppose that $\{e_1, \dots, e_{d/2}, f_1, \dots, f_{d/2}\}$ is a symplectic basis and each $U_i = \langle e_i, f_i \rangle$. Let $\gamma \in \Omega$ be the decomposition $V = \langle e_1, f_1 + f_2 \rangle \perp \langle e_1 + e_2, f_1 \rangle \perp \langle e_3, f_3 \rangle \perp \dots \perp \langle e_d, f_d \rangle$. Then H_γ fixes $\{\langle e_1 \rangle, \langle f_1 \rangle\}$ while $\text{Sp}_2(q) \leq H_{\langle e_1, f_1 \rangle}^{(e_1, f_1)}$. Thus G has a p -subdegree and as $|\Omega|$ is divisible by p , G is not $\frac{3}{2}$ -transitive.

6.2.6. $m = 2$ and G is orthogonal: Here H preserves the decomposition $V = U_1 \perp U_2 \perp \dots \perp U_{d/2}$ and the U_i are isometric nondegenerate 2-spaces, either all hyperbolic or all elliptic. If each U_i is hyperbolic then V is hyperbolic and so has a hyperbolic basis $\{e_1, \dots, e_{d/2}, f_1, \dots, f_{d/2}\}$ and we may suppose that each $U_i = \langle e_i, f_i \rangle$. On the other hand, if each U_i is elliptic, let $U_i = \langle a_i, b_i \rangle$ such that $Q(a_i) = 1, Q(b_i) = \zeta$ for some $\zeta \in \text{GF}(q) \setminus \{0\}$, and $B(a_i, b_i) = 0$ if q is odd and $B(a_i, b_i) = 1$ if q is even. Note that if p divides $|H|$ either $p \leq d/2$ or H contains a field automorphism of order p . Moreover, $d/2 \geq 4$.

Suppose first that q is even. If each U_i is hyperbolic, let $W_1 = \langle e_1 + e_2, f_1 \rangle$ and $W_2 = \langle e_2, f_1 + f_2 \rangle$. Then $\gamma = W_1 \perp W_2 \perp U_3 \perp \dots \perp U_{d/2} \in \Omega$ and H_γ fixes U_1 and U_2 . Thus $2^{\binom{d/2}{2}}$ divides $|\gamma^H|$ and so G has an even subdegree. If each U_i is elliptic, then

$$\gamma = \langle a_1, b_1 + a_2 + a_3 \rangle \perp \langle a_2 + b_1 + b_3, b_2 \rangle \perp \langle a_3, b_3 + b_1 + b_2 \rangle \perp U_4 \perp \dots \perp U_{d/2} \in \Omega$$

and H_γ fixes U_1, U_2 and U_3 . Thus $|\gamma^H|$ is divisible by $6^{\binom{d/2}{3}}$ which is even.

Suppose now that q is odd and $p \leq d/2$. By [34, Propositions 6.2.9 and 6.2.10], we have $q \geq 5$. If each U_i is hyperbolic, let $W_1 = \langle e_1, f_1 + f_3 \rangle$, $W_2 = \langle e_1 + e_2 - e_3, f_2 \rangle$ and $W_3 = \langle e_1 - e_3, -f_2 - f_3 \rangle$. If each U_i is elliptic, since $q \geq 5$, we can choose $\lambda_1, \lambda_2 \in \text{GF}(q) \setminus \{0\}$ such that $Q(\lambda_1 a_1 + \lambda_2 b_2) = 1$ and then there exists $\mu_1, \mu_2 \in \text{GF}(q) \setminus \{0\}$ such that $Q(\mu_1 a_1 + \mu_2 b_2) = \zeta$ and $B(\lambda_1 a_1 + \lambda_2 b_2, \mu_1 a_1 + \mu_2 b_2) = 0$. Then let $W_1 = \langle \lambda_1 a_1 + \lambda_2 b_2, b_1 \rangle$, $W_2 = \langle \mu_1 a_1 + \mu_2 b_2, a_3 \rangle$ and $W_3 = \langle a_2, b_3 \rangle$. Now for both the hyperbolic and elliptic case let $\beta \in \Omega$ be the decomposition $V = W_1 \perp W_2 \perp W_3 \perp U_4 \perp \dots \perp U_{d/2}$. Then H_β fixes U_1, U_2 and U_3 , so $6^{\binom{d/2}{3}}$ divides $|\beta^H|$. Thus if $p = 3$ then G has a subdegree divisible by p . For $p \geq 5$, note that for each pair $U_i \perp U_{i+1}$, with $4 \leq i \leq p-1$ and i even, we can find Y_i, Y_{i+1} of the same isometry type as U_1 such that $Y_i \perp Y_{i+1} = U_i \perp U_{i+1}$ and $\{Y_i, Y_{i+1}\} \cap \{U_i, U_{i+1}\} = \emptyset$. Let γ be the decomposition $V = W_1 \perp W_2 \perp W_3 \perp Y_4 \perp Y_5 \perp \dots \perp Y_{p-1} \perp Y_p \perp U_{p+1} \perp \dots \perp U_{d/2}$. Then H_γ fixes $\{U_1, U_2, U_3\}$ and $\{U_1, U_2, \dots, U_p\}$, so $|\gamma^H|$ is divisible by $\binom{p}{3} \binom{d/2}{p}$, which is divisible by p .

Suppose now that $p > d/2$ and H contains a field automorphism x of order p . Note that since $d \geq 8$ this implies that $p \geq 5$ and $q = p^f \geq p^p$. By Lemma 5.8, $|x^G| > \frac{1}{4} q^{d(d-1)/5}$. Now $|H| \leq (2(q+1))^{d/2} (d/2)! (q-1) f 3 < q^{3d/4+3}$ due to the conditions on q . Thus $|x^G \cap H|^2 < |x^G|$ and so by Lemma 2.6, G has a subdegree divisible by p .

Now since p divides $|\Omega|$ it follows that if p divides $|H|$ then G is not $\frac{3}{2}$ -transitive. If p does not divide $|H|$ note that $p \geq 5$. As above, choose Y_1, Y_2 to be nondegenerate 2-subspaces of $\langle U_1, U_2 \rangle$ of the same isometry type as U_1 and orthogonal to each other. Let $\beta' \in \Omega$ be the decomposition given by $V = Y_1 \perp Y_2 \perp U_3 \perp \dots \perp U_{d/2}$. Then $\binom{d/2}{2}$ divides $|(\beta')^H|$ and so let ℓ be the integer such that $|(\beta')^H| = \ell \binom{d/2}{2}$. Now let Y_3, Y_4 be nondegenerate 2-subspaces of $\langle U_3, U_4 \rangle$ of the same isometry type as U_1 and orthogonal to each other. Then we can choose Y_3, Y_4 so that if $\beta'' \in \Omega$ is the decomposition $V = U_1 \perp U_2 \perp Y_3 \perp Y_4 \perp U_5 \perp \dots \perp U_{d/2}$ then $|(\beta'')^H| = \ell \binom{d/2}{2}$. Now let γ be the decomposition $V = Y_1 \perp Y_2 \perp Y_3 \perp Y_4 \perp U_5 \perp \dots \perp U_{d/2}$. Then H_γ fixes $\{U_1, U_2, U_3, U_4\}$ and preserves the partition $\{\{U_1, U_2\}, \{U_3, U_4\}\}$. Thus $|\gamma^H| = 3\ell^2 \binom{d/2}{4}$. Except in the case where $d/2 = 4$ and $\ell \neq 2$, we have $|\gamma^H| \neq |(\beta')^H|$ and so G is not $\frac{3}{2}$ -transitive. When $d/2 = 4$ and $\ell = 2$, note that $|\gamma^H| = 12$ while for β introduced previously (and whose definition is still valid for $p > d/2$), $|\beta^H|$ is divisible by $6 \binom{4}{3} = 24$. Hence G is not $\frac{3}{2}$ -transitive in this case either.

6.2.7. $m = 1$ and G is orthogonal: According to [34, Table 4.2A], $q = p$ is odd, and so if p divides $|H|$ it follows that $p \leq d$. Recall that we are assuming $d \geq 7$. Let H be the stabiliser of the decomposition $\alpha = \langle v_1 \rangle \perp \langle v_2 \rangle \perp \dots \perp \langle v_d \rangle$. By the discussion in [34, p100-101] we may assume that $B(v_i, v_i) = 1$ for all i and hence $Q(v_i) = 2^{-1}$. Note also that not all orthogonal decompositions of V into nonsingular 1-spaces may lie in Ω as there are two isometry types of nonsingular 1-spaces. Note also that for $\lambda \in \text{GF}(p)$, $Q(v_1 + \lambda v_2) = Q(v_1)(1 + \lambda^2)$ and so $Q(v_1 + \lambda v_2) = 0$ if and only if $\lambda^2 = -1$. Hence the space

$$\langle v_1, v_2 \rangle \text{ is } \begin{cases} \text{elliptic when } q \equiv 3 \pmod{4} \\ \text{hyperbolic when } q \equiv 1 \pmod{4} \end{cases}$$

Suppose first that p divides $|H|$ with $p > 5$. Then there exist at least three 1-dimensional subspaces of $\langle v_1, v_2 \rangle$ upon which the quadratic form has the same parity as it has on $\langle v_1 \rangle$. Thus there exist $\lambda_1, \lambda_2 \in \text{GF}(p) \setminus \{0\}$ such that $Q(\lambda_1 v_1 + \lambda_2 v_2) = Q(v_1)$ and hence there exists $g \in L$ mapping v_1 to

$\lambda_1 v_1 + \lambda_2 v_2$. Now $\langle \lambda_1 v_1 + \lambda_2 v_2 \rangle^\perp \cap \langle v_1, v_2, v_3 \rangle = \langle \mu_1 v_1 + \mu_2 v_2, v_3 \rangle$ for some $\mu_1, \mu_2 \neq 0$, and this subspace has a decomposition $\langle x_1 \rangle \perp \langle x_2 \rangle$ with $Q(x_1) = Q(x_2) = Q(v_1)$ such that each $x_i \neq v_3$. Note that each $x_i = \sum_{j=1}^3 \xi_j v_j$ with each $\xi_j \neq 0$. Now

$$\gamma = \langle \lambda_1 v_1 + \lambda_2 v_2 \rangle \perp \langle x_1 \rangle \perp \langle x_2 \rangle \perp \langle \lambda_1 v_4 + \lambda_2 v_5 \rangle \perp \langle \mu_1 v_4 + \mu_2 v_5 \rangle \perp \dots \perp \langle \lambda_1 v_{p-1} + \lambda_2 v_p \rangle \perp \langle \mu_1 v_{p-1} + \mu_2 v_p \rangle \perp \langle v_{p+1} \rangle \perp \dots \perp \langle v_d \rangle \in \Omega.$$

Then H_γ fixes $\langle v_3 \rangle$, $\{\langle v_1 \rangle, \langle v_2 \rangle\}$ and $\{\langle v_1 \rangle, \dots, \langle v_p \rangle\}$ and so $|\gamma^H|$ is divisible by $3 \binom{p}{3} \binom{d}{p}$, which is divisible by p .

Suppose next that $p = 5$. Since $d \geq 7$, $|H|$ is divisible by 5. In this case $Q(v_1) = 2^{-1} = 3$, which is a nonsquare. Hence if $Q(x)$ is a nonsquare then $\langle x \rangle \in \langle v_1 \rangle^G$. The nonsquares in $\text{GF}(5)$ are 2 and 3. Let $x_1 = v_1 + 2v_2 + v_3$, $x_2 = v_1 + v_2 + 2v_3$, $x_3 = 2v_1 + v_2 + v_3 + v_4 + 2v_5$, $x_4 = 2v_1 + v_2 + v_3 + 3v_4 + 3v_5$ and $x_5 = 2v_1 + v_2 + v_3 + 2v_4 + v_5$. Then $Q(x_1) = Q(x_2) = Q(x_3) = Q(x_5) = 3$ and $Q(x_4) = 2$, and so

$$\delta = \langle x_1 \rangle \perp \langle x_2 \rangle \perp \langle x_3 \rangle \perp \langle x_4 \rangle \perp \langle x_5 \rangle \perp \langle v_6 \rangle \perp \dots \perp \langle v_d \rangle \in \alpha^G.$$

Now H_δ fixes $\{\langle v_1 \rangle, \langle v_2 \rangle, \langle v_3 \rangle\}$ and $\{\langle v_4 \rangle, \langle v_5 \rangle\}$. Hence $10 \binom{d}{5}$ divides $|\delta^H|$.

Next suppose that $p = 3$. Since $d \geq 7$, $|H|$ is divisible by 3. In this case $Q(v_1) = 2^{-1} = 2$, which is a nonsquare. Let $x_1 = v_1 + v_2 + v_3 + v_4$, $x_2 = v_1 - v_2 + v_5 + v_6$, $x_3 = v_3 - v_4 + v_5 - v_6$, $x_4 = v_1 + v_2 - v_3 - v_4$, $x_5 = -v_1 + v_2 + v_5 + v_6$ and $x_6 = -v_3 + v_4 + v_5 - v_6$. Then $Q(x_i) = 2$ for all i , and so

$$\delta = \langle x_1 \rangle \perp \langle x_2 \rangle \perp \langle x_3 \rangle \perp \langle x_4 \rangle \perp \langle x_5 \rangle \perp \langle x_6 \rangle \perp \langle v_7 \rangle \perp \dots \perp \langle v_d \rangle \in \alpha^G.$$

Now H_δ preserves that partition $\{\{\langle v_1 \rangle, \langle v_2 \rangle\}, \{\langle v_3 \rangle, \langle v_4 \rangle\}, \{\langle v_5 \rangle, \langle v_6 \rangle\}\}$. Hence $15 \binom{d}{6}$ divides $|\delta^H|$.

Thus if p divides $|H|$ we have found a p -subdegree and so as p divides $|\Omega|$, it follows that G is not $\frac{3}{2}$ -transitive. Suppose now that p does not divide $|H|$. Then $p > d$ and in particular $p > 7$. Using $\lambda_1, \lambda_2, \mu_1$ and μ_2 as before let

$$\beta = \langle \lambda_1 v_1 + \lambda_2 v_2 \rangle \perp \langle \mu_1 v_1 + \mu_2 v_2 \rangle \perp \langle v_3 \rangle \perp \dots \perp \langle v_d \rangle \in \Omega.$$

Now H_β fixes $\{\langle v_1 \rangle, \langle v_2 \rangle\}$ and hence $|\beta^H| = 2^i \binom{d}{2}$ for some $i \geq 0$. Next let

$$\eta = \langle \lambda_1 v_1 + \lambda_2 v_2 \rangle \perp \langle \mu_1 v_1 + \mu_2 v_2 \rangle \perp \langle \lambda_1 v_3 + \lambda_2 v_4 \rangle \perp \langle \mu_1 v_3 + \mu_2 v_4 \rangle \perp \langle v_5 \rangle \perp \dots \perp \langle v_d \rangle \in \Omega.$$

Then H_η fixes $\{\{\langle v_1 \rangle, \langle v_2 \rangle\}, \{\langle v_3 \rangle, \langle v_4 \rangle\}\}$. Hence $|\eta^H|$ is divisible by $3 \binom{d}{4} = \binom{d}{2} (d-2)(d-3)/4$. One of $d-2$ or $d-3$ is an odd number at least 5, and hence $|\eta^H| \neq |\beta^H|$. Thus G is not $\frac{3}{2}$ -transitive.

6.2.8. $m = 1$ and G is unitary: Let H be the stabiliser of the decomposition $V = \langle v_1 \rangle \perp \langle v_2 \rangle \perp \dots \perp \langle v_d \rangle$. By the discussion on [34, p100–101] we may assume that $B(v_i, v_i) = 1$ for all i and by [34, Proposition 4.2.9], $H \cap L = ((q+1)^{d-1}/(q+1, d)).S_d$. Thus if p divides $|H|$ either $p \leq d$ or H contains a field automorphism of order p . Suppose first that $p = 2$. Then $B(v_1 + v_2 + v_3, v_1 + v_2 + v_3) = 1$ and $\langle v_1, v_2, v_3 \rangle \cap \langle v_1 + v_2 + v_3 \rangle^\perp = \langle v_1 + v_2, v_2 + v_3 \rangle$. Both $v_1 + v_2$ and $v_2 + v_3$ are singular. Since $\langle v_1 + v_2, v_2 + v_3 \rangle$ is nondegenerate, there exist nonsingular $x_1, x_2 \in \langle v_1 + v_2, v_2 + v_3 \rangle$ with $x_1 \in x_2^\perp$. Note that for $i = 1, 2$, $x_i \neq v_1, v_2, v_3$. Thus

$$\beta = \langle v_1 + v_2 + v_3 \rangle \perp \langle x_1 \rangle \perp \langle x_2 \rangle \perp \langle v_4 \rangle \perp \dots \perp \langle v_d \rangle \in \Omega.$$

Moreover, H_β fixes $\langle v_1, v_2, v_3 \rangle$. Hence $\binom{d}{3} = d(d-1)(d-2)/6$ divides $|\beta^H|$. If d is even, or $d \equiv 1 \pmod{4}$, it follows that G has an even subdegree. If $d \equiv 3 \pmod{4}$ and $d > 3$, let

$$\gamma = \langle v_1 + v_2 + v_3 \rangle \perp \langle x_1 \rangle \perp \langle x_2 \rangle \perp \langle v_4 + v_5 + v_6 \rangle \perp \langle y_1 \rangle \perp \langle y_2 \rangle \perp \langle v_7 \rangle \perp \dots \perp \langle v_d \rangle \in \Omega,$$

where y_1, y_2 are nonsingular elements of $\langle v_4 + v_5, v_5 + v_6 \rangle$. Then $|\gamma^H|$ is divisible by $10 \binom{d}{6}$, which is even. Hence if $d > 3$, G has a subdegree divisible by $p = 2$. Now let $d = 3$ and let $x \in H \cap L$ be an involution. Then by Lemma 5.9, $|x^G| = (q-1)(q^3+1)$ while $|x^G \cap H| = 3(q+1)$. Hence for $q \geq 8$, Lemma 2.6 implies that G has an even subdegree. If $q = 4$, a MAGMA calculation shows that L has a subdegree equal to 150. Hence for all even $q \geq 4$, G has an even subdegree (note that $(d, q) \neq (3, 2)$).

Next suppose that p is odd and $p \leq d$. In $\langle v_1, v_2, v_3 \rangle$, we have $\langle v_1 + v_2 \rangle^\perp = \langle v_1 - v_2, v_3 \rangle$. Then there exist $\lambda, \mu \in \text{GF}(q^2) \setminus \{0\}$ such that $v_1 - v_2 + \lambda v_3$ and $v_1 - v_2 + \mu v_3$ are nonsingular and $B(v_1 - v_2 + \mu v_3, v_1 - v_2 + \lambda v_3) = 0$. Then

$$(1) \quad \gamma = \langle v_1 + v_2 \rangle \perp \langle v_1 - v_2 + \lambda v_3 \rangle \perp \langle v_1 - v_2 + \mu v_3 \rangle \perp \langle v_4 \rangle \perp \dots \perp \langle v_d \rangle \in \Omega.$$

Now H_γ fixes $\{\langle v_1 \rangle, \langle v_2 \rangle\}$ and $\langle v_3 \rangle$, and if $g \in H_\gamma$ maps v_1 to ξv_1 then g maps v_2 to ξv_2 . Hence $3(q+1)\binom{d}{3}$ divides $|\gamma^H|$. Thus if $p = 3$ we have found a p -subdegree. If $p \geq 5$, let

$$\begin{aligned} \delta = & \langle v_1 + v_2 \rangle \perp \langle v_1 - v_2 + \lambda v_3 \rangle \perp \langle v_1 - v_2 + \mu v_3 \rangle \perp \langle v_4 + v_5 \rangle \perp \\ & \langle v_4 - v_5 \rangle \perp \dots \perp \langle v_{p-1} + v_p \rangle \perp \langle v_{p-1} - v_p \rangle \perp \langle v_{p+1} \rangle \perp \dots \perp \langle v_d \rangle \in \Omega. \end{aligned}$$

Then $\binom{p}{3}\binom{d}{p}$ is divisible by p and divides $|\delta^H|$.

Finally, suppose that $p > d$ and H contains a field automorphism x of order p . Note that $p \geq 5$ and $q \geq p^p > d!$. By Lemma 5.8, $|x^G| > \frac{q}{2(q+1)}q^{(4d^2-9)/5} > q^{(4d^2-10)/5}$, as $4 < q^{1/5}$. Now $|H| \leq (q+1)^{d-1}d!2f < (q+1)^{d-1}q^2 < (2q)^{d-1}q^2 < q^{(4d+2)/3}$. Hence $|x^G| > |x^G \cap H|^2$ for $d \geq 5$. For $d = 4$, we see that $|H| \leq (q+1)^3 48f < q^3 384f < q^4$ as $q \geq 5^5$. Since $|x^G| > q^{10}$ we also have $|x^G| > |x^G \cap H|$ in this case. Finally, for $d = 3$ we have $|H| \leq (q+1)^2 12f < \frac{11}{10}q^2 12f < q^{13/5}$ as both p and f are at least 5. Since $|x^G| > q^{26/5}$ in this case we also have that $|x^G| > |x^G \cap H|^2$. Hence for all values of d , Lemma 2.6 yields a p -subdegree.

Now p divides $|\Omega|$ and so if p divides $|H|$, G is not $\frac{3}{2}$ -transitive. If $(|H|, p) = 1$ then p is odd and $p > d$. Let $\beta = \langle v_1 + v_2 \rangle \perp \langle v_1 - v_2 \rangle \perp \langle v_3 \rangle \perp \dots \perp \langle v_d \rangle \in \Omega$. If $g \in H_\beta$ then g can interchange v_1 and v_2 , but if $g : v_1 \mapsto \lambda v_1$ then $g : v_2 \mapsto \lambda v_2$. Thus $|\beta^H| = (q+1)\binom{d}{2}$. For γ defined in (1) we have already seen that $|\gamma^H|$ is divisible by $3(q+1)\binom{d}{3}$ (the definition of γ and $|\gamma^H|$ do not depend on the condition $p \leq d$). Thus if $d > 3$, then G is not $\frac{3}{2}$ -transitive. If $d = 3$, we see that $|H_\gamma \cap L| = 2$ and so $3(q+1)^2/(q+1, 3)$ divides $|\gamma^H|$. Since $(d, q) \neq (3, 2)$, it follows that $|\gamma^H| \neq |\beta^H|$ and so G is not $\frac{3}{2}$ -transitive.

6.3. Aschbacher class \mathcal{C}_3 : Here H is the stabiliser of an extension field structure of V as a b -dimensional vector space over $\text{GF}(q^a)$ where $d = ab$ and a is prime. The subgroups are described in detail in [34, §4.3].

Lemma 6.1. *If $b > 1$ then G has a p -subdegree and G is not $\frac{3}{2}$ -transitive.*

Proof. Suppose first that if G is orthogonal then H is not unitary. By hypothesis, $b \geq 2$ and if G is orthogonal, [34, Table 4.3.A] implies that $b \geq 3$. Since $a \neq 1$ it follows that H is insoluble. Let T be a $\text{GF}(q)$ -subfield subgroup of H . Choose a $\text{GF}(q^a)$ basis for V and let W be the $\text{GF}(q)$ -span of this basis. Then we can choose T to preserve the decomposition $V = W \otimes \text{GF}(q^a)$. Since $\text{GF}(q^a)$ is a -dimensional over $\text{GF}(q)$, we can find an element of $G \setminus H$ which preserves this tensor decomposition and centralises T . Hence $N_H(T) < N_G(T)$. Let S be a Sylow p -subgroup of H . Then $\langle S, T \rangle$ contains the weakly closed normal subgroup $H^{(\infty)}$ of H and so Lemma 2.2 implies that G has a p -subdegree.

Next we suppose $L = \text{P}\Omega_{2d}^\pm(q)$ and H is a unitary group of dimension d . By Proposition 5.2, $H_0 = H^{(\infty)}$ has a cyclic subgroup T such that for $S \in \text{Syl}_p(H)$, we have $H_0 \leq \langle S, T \rangle$. Now T fixes a nondegenerate $\text{GF}(q^2)$ -hyperplane and acts trivially on its perp. Thus T fixes a $\text{GF}(q)$ -subspace of codimension 2 and acts trivially on its perp. Hence $N_H(T) < N_G(T)$ and so Lemma 2.2 implies that G has a subdegree divisible by p .

Since $|\Omega|$ is always divisible by p and G has a p -subdegree, it follows that G is not $\frac{3}{2}$ -transitive. \square

It remains to deal with the case where $b = 1$. This only occurs where $L = \text{PSL}_d^\epsilon(q)$ and d is prime. First we deal with the case where $(d, q) = (2, 2^f)$. Recall that a transitive group is strongly $\frac{3}{2}$ -transitive if all non-principal constituents of the permutation character are distinct and have the same degree.

Lemma 6.2. *Let G be a group with socle $\text{PSL}_2(q)$, where $q = 2^f \geq 4$, and let H be a maximal subgroup of G with $H \cap L$ dihedral of order $2(q+1)$. Consider the action of G of degree $q(q-1)/2$ on the set of cosets of H .*

- (1) *G has no even subdegrees if and only if the index $|G : L|$ is odd.*
- (2) *G is strongly $\frac{3}{2}$ -transitive if and only if either $G = L$ or f is prime.*
- (3) *G is $\frac{3}{2}$ -transitive if and only if either $G = L$ or f is prime.*
- (4) *G is 2-transitive if and only if $q = 4$.*

The nontrivial character degrees and the nontrivial subdegrees of the examples in (2) and (3) are $q+1$ when $G = L$ and $(q+1)f$ when $G = L.f$.

Proof. We work with characters of L – the character theory for the groups $\text{PSL}_2(q)$ is very well known, and so is the action of the outer automorphisms on the irreducible characters (see, eg, [24, p500]). We supply details for completeness.

There are precisely $k = (q-2)/2$ irreducible characters of L of degree $q+1$. We denote these by χ_j . The permutation character of our action of L is $\pi = 1 + \chi_1 + \dots + \chi_k$: to check that the multiplicity of

each χ_j is 1, it is enough by Frobenius reciprocity to check that the sum of $\chi_j(h)$ over $h \in H \cap L$ equals $2(q+1)$; this is clear, since the value of χ_j on the $q+1$ involutions is 1, and on all nontrivial elements of order dividing $q+1$ is 0. The assertion now follows by comparing the degrees of π and the character sum on the right. It now follows that L is always strongly $\frac{3}{2}$ -transitive, and hence $\frac{3}{2}$ -transitive, with rank $k+1$ and the nontrivial subdegrees are all $q+1$.

We now consider the outer automorphisms of L . The outer automorphism group is cyclic of order f , and we can choose the representative σ of the generating coset of the group of inner automorphisms in the automorphism group in the usual way as the generator of the group of automorphisms induced by the Galois group of the field $\text{GF}(q)$. The elements of L that we have not mentioned yet have orders dividing $q-1$ and are conjugate to powers of the diagonal matrix a with diagonal entries ρ, ρ^{-1} , where ρ is a primitive element of $\text{GF}(q)$. Then $\chi_j(a^i) = \epsilon^{ij} + \epsilon^{-ij}$, where ϵ is a primitive complex $(q-1)^{\text{th}}$ root of unity. The action of σ takes a to a^2 , and its action on the set $\{\chi_1, \dots, \chi_k\}$ is equivalent to the action of the generator of the Galois group sending ρ to ρ^2 .

All the assertions now follow easily, noting in addition that the number of orbits of any outer automorphism of L on the set of nontrivial $(H \cap L)$ -orbits in this action equals the number of orbits on the set $\{\chi_1, \dots, \chi_k\}$.

(1) If $|G:L|$ is odd, all subdegrees of G are odd, since this is true of L and the outer automorphisms present in G have odd order. Hence G has no even subdegrees.

If $|G:L|$ is even, an involutory outer automorphism in G will act nontrivially on the set $\{\chi_1, \dots, \chi_k\}$ and hence will fuse two of the L -suborbits, so G will have an even subdegree.

(2) and (3). If f is prime, σ will act semiregularly on the set $\{\chi_1, \dots, \chi_k\}$, whence G is strongly $\frac{3}{2}$ -transitive and hence also $\frac{3}{2}$ -transitive.

(4) When $q=4$ we have $\text{PSL}_2(4) \cong \text{PSL}_2(5)$ and the action is 2-transitive of degree 6.

Conversely, if G is $\frac{3}{2}$ -transitive, then σ must be semiregular on the set of $(H \cap L)$ -orbits, whence it must be semiregular also on the set $\{\chi_1, \dots, \chi_k\}$, and hence as a Galois automorphism of $\text{GF}(q)$. It follows that f is a prime. \square

We remark that it is not hard to find an explicit G -invariant correspondence between the set of orbitals of the action of L on the sets of H -cosets and $\text{GF}(q)$, making the action of σ quite explicit. This was done in much more generality by Inglis [32] (note that $\text{PSL}_2(q) = \text{Sp}_2(q)$ and the maximal dihedral subgroups of $\text{PSL}_2(q)$ are the orthogonal groups $O_2^\pm(q)$).

Before dealing with the remaining cases we need to set up some notation. By [34, Proposition 4.3.6], $H \cap L = C_\ell \rtimes C_d$ where $\ell = (q^d - \epsilon)/((q - \epsilon)(d, q - \epsilon))$. Suppose first that $\epsilon = +$. Then we can identify V with $\text{GF}(q^d)$. Let μ be a primitive element of $\text{GF}(q^d)$. Define the maps $\mu : \xi \mapsto \xi\mu$ and $\phi : \xi \mapsto \xi^q$ on V . Both are $\text{GF}(q)$ -linear and $\langle \mu, \phi \rangle \cong C_{q^d-1} \rtimes C_d$. Then $H \cap L$ is the image in $\text{PSL}_d(q)$ of the intersection of $\langle \mu, \phi \rangle$ with $\text{SL}_d(q)$. Note that $\xi^\phi = \xi$ if and only if $\xi \in \text{GF}(q)$ and so 1 is an eigenvalue of ϕ with multiplicity 1. For $\epsilon = -$ we have $d \geq 3$, and we identify V with $\text{GF}(q^{2d})$ and let $\mu \in \text{GF}(q^{2d})$ have order $q^d + 1$. We then define the maps $\mu : \xi \mapsto \xi\mu$ and $\phi : \xi \mapsto \xi^{q^2}$ on V which are both $\text{GF}(q^2)$ -linear and preserve the Hermitian form $B(\xi_1, \xi_2) = \text{Tr}_{q^{2d} \rightarrow q^d}(\xi_1 \xi_2^{q^d})$. Then $\langle \mu, \phi \rangle \cong C_{q^d+1} \rtimes C_d$ and $H \cap L$ is the image in $\text{PSU}_d(q)$ of the intersection of $\langle \mu, \phi \rangle$ with $\text{SU}_d(q)$. Moreover, 1 is an eigenvalue of ϕ with multiplicity 1. Indeed, for both values of ϵ , $x^d - 1$ is the characteristic polynomial of ϕ and so for $d \neq p$, each d^{th} root of unity occurs as an eigenvalue with multiplicity 1. When $d = p$, the element ϕ has Jordan form J_p .

We first look for p -subdegrees. Recall Definition 5.6 of $\nu(x)$ for an element $x \in \text{PGL}_d(q)$.

Lemma 6.3. *Let G be an almost simple group with socle $L = \text{PSL}_d^\epsilon(q)$ and H be as above with $(d, q) \neq (2, 2^f)$. If p divides $|H|$ and G has no subdegrees divisible by p then G is the 2-transitive group $L_3(2).2$ of degree 8 as in line 7 of Table 3.*

Proof. Suppose first that $d \geq 3$ and that p divides $|H|$. Then either $p = d$ or H contains a field automorphism, graph automorphism or graph-field automorphism of order p . If $p = d$ then p is odd and we let $x = \phi \in H \cap L$ have order p . Since ϕ has Jordan form J_p , we have $\nu(x) = d-1$. Thus by Proposition 5.7, $|x^G| > \frac{q}{2(q-\epsilon)(q+1)} q^{d(d-1)} > q^{d(d-1)-3}$. Since $|H \cap L| < q^{d+1}$ it follows from Lemma 2.6(ii) that if $p = d \geq 5$ then G has a subdegree divisible by p . For $d = p = 3$, [9, Lemma 3.18 and Lemma 3.20] implies that $|x^L| = q(q^2 - 1)(q^3 - \epsilon)$ while we have that $|x^L \cap H| \leq 2(q^2 + \epsilon q + 1)$. Thus except for $(q, \epsilon) = (3, +)$, Lemma 2.6(ii) yields a subdegree divisible by $p = 3$. For $q = 3$, a MAGMA [7] calculation shows that the subdegrees for $L = \text{PSL}_3(3)$ are $\{1, 13^5, 39^2\}$. Thus we may assume that $d \neq p$ and H contains a field, graph or graph-field automorphism x of order p . Now $N_{\text{Aut}(L)}(H) = C_{(q^d-1)/(q-1)} \rtimes (C_{df} \times C_2)$ or $C_{(q^d+1)/(q+1)} \rtimes C_{2df}$ for $\epsilon = +$ or $-$ respectively. It follows that $|x^G \cap H| \leq q^d$. Now by Lemma 5.8,

$|x^G| > \frac{1}{2}q^{(d^2-3)/2}$. Thus for $d \geq 5$, $|x^G| > |x^G \cap H|^2$ and so by Lemma 2.6(ii), G has a p -subdegree. This leaves the cases $d = 3$ and $d = 2$.

Let $d = 3$. Then $p = 2$ or $p \geq 5$. Let $\overline{H} = N_{\text{PGL}_3(q)}(H)$. For $p \geq 5$, we have that x is a field automorphism and so Lemma 5.8 implies that $|x^G| > \frac{1}{2} \left(\frac{q}{q+1} \right)^{(1-\epsilon)/2} q^{(4d^2-9)/5} > q^{22/5} > q^4$. Now $C_{\overline{H}}(x) = C_{q^{2/p} + \epsilon q^{1/p} + 1} \rtimes C_3$ and there is only one conjugacy class of subgroups of H of order p . Thus $|x^G \cap H| < (p-1)(q^2 + \epsilon q + 1)/(q^{2/p} + \epsilon q^{1/p} + 1) < (p-1)2q^{2-2/p} < q^2$. Hence $|x^G| > |x^G \cap H|^2$ and Lemma 2.6(ii) yields a p -subdegree. So we now assume that $p = 2$. If x is a field automorphism of order 2 then $\epsilon = +$ and so Lemma 5.8 implies $|x^G| > \frac{1}{2}q^3$ while we have $|x^G \cap H| < (q^2 + q + 1)/(q + q^{1/2} + 1) < q$. Thus Lemma 2.6(ii) again gives a p -subdegree. If x is a graph-field automorphism then $\epsilon = +$ and by [34, Proposition 4.8.5], $C_L(x) \leq \text{PGU}_3(q^{1/2})$. Hence $|x^G| \geq |L|/|\text{PGU}_3(q^{1/2})| = q^{3/2}(q^{3/2}-1)(q+1)/(3, q-1)$. Moreover, $C_{\overline{H}}(x) = C_{q^{3/2}+1} \rtimes C_3$. Hence $|x^G \cap H|^2 \leq (q^{3/2}-1)^2 < |x^G|$ and so by Lemma 2.6(ii), G has a subdegree divisible by p . Finally, let x be a graph automorphism of order 2. By [4, (19.9)], $C_L(x) = \text{PSO}_3(q)$ and so $|x^L| = q^2(q^3 - \epsilon)/(3, q - \epsilon)$. Moreover, $|x^G \cap H| < q^2 + \epsilon q + 1$ and so for $q \geq 8$ or $(q, \epsilon) = (4, -)$, Lemma 2.6(ii) implies that G has a subdegree divisible by p . This leaves us to consider $G = \text{PSL}_3(2).2$ of degree 8, and primitive groups of degree 960 with socle $L = \text{PSL}_3(4)$ (note that $(q, \epsilon) \neq (2, -)$). The first has subdegrees 1 and 7 and so does not have an even subdegree. A MAGMA [7] calculation shows that all groups in the second case with an even order point stabiliser have an even subdegree.

Finally we deal with the case where $d = 2$ and so $L = \text{PSL}_2(q)$. Recall that $q = p^f$ is odd. If p divides $|H|$ then H contains a field automorphism x of order p and in particular, p divides f . Then $C_L(x) = \text{PSL}_2(q^{1/p})$ and so $|x^G| > q^{3-3/p} \geq q^2$. Moreover, letting $\overline{H} = N_{\text{Aut}(L)}(H)$ we have $|\overline{H}| = 2(q+1)f$ and $|C_{\overline{H}}(x)| = 2(q^{1/p} + 1)f$. Since the Sylow p -subgroup of \overline{H} is cyclic, it contains a unique conjugacy class of subgroups of order p and so $|x^G \cap H| \leq (q^{1-1/p} - q^{1-2/p} + \dots + 1)(p-1) < q$. Thus Lemma 2.6(ii) implies that G has a subdegree divisible by p . \square

To deal with $\frac{3}{2}$ -transitivity, we first need the following lemma.

Lemma 6.4. *Let p and d be primes with d odd and let $q = p^f$ for some positive integer f . Then d does not divide $y = (q^d - \epsilon)/((q - \epsilon)(q - \epsilon, d))$ for $\epsilon = \pm 1$*

Proof. By Fermat's Little Theorem, d divides $q^{d-1} - 1$. Suppose that d divides y . Then d divides $q^d - \epsilon$ and so divides $q^d - \epsilon q^{d-1} = q^{d-1}(q - \epsilon)$. Since d divides y it is coprime to q and so d divides $q - \epsilon$. If $q \equiv ad + \epsilon \pmod{d^2}$ then using the Binomial expansion we see that for each r , $q^r \equiv (\epsilon)^r + rad(\epsilon)^{r-1} \pmod{d^2}$. Hence $(q^d - \epsilon)/(q - \epsilon) \equiv d + \epsilon ad + \epsilon 2ad + \dots + \epsilon(d-1)ad \pmod{d^2} \equiv d + \epsilon ad(d-1)/2 \pmod{d^2} \equiv d \pmod{d^2}$. Thus d does not divide $(q^d - \epsilon)/((q - \epsilon)(q - \epsilon, d))$. \square

Lemma 6.5. *Let G be an almost simple group with socle $L = \text{PSL}_d^\epsilon(q)$ and H be as above with $(d, q) \neq (2, 2^f)$. Then G is not $\frac{3}{2}$ -transitive unless $L = \text{PSL}_3(2)$ acting 2-transitively of degree 8.*

Proof. Since p divides $|\Omega|$ in all cases, if G has a p -subdegree then G is not $\frac{3}{2}$ -transitive. Thus by Lemma 6.3, we are left to consider the case where p does not divide $|H|$. In particular, if $d \geq 3$ then $p \neq d$. We note that if $(d, \epsilon, q) = (3, +, 2)$ then $L \cong \text{PSL}_2(7)$ acting 2-transitively of degree 8, so assume we are not in this case.

Suppose first that $d \geq 3$ with $p \neq d$. Now for $d \geq 3$, we have $H \cap L = C_\ell \rtimes C_d$ where $\ell = (q^d - \epsilon)/((q - \epsilon)(q - \epsilon, d))$. Note that for $M \leq C_\ell$ with M nontrivial, Lemma 6.4 implies that $|M|$ is coprime to d and so as $H = N_G(M)$ it follows that if $M \leq H \cap H^g$ then $g \in H$. Thus for all $g \in G \setminus H$, either $H \cap H^g \cap L = C_d$ or $H \cap H^g \cap L = 1$. Now for $d \neq p$, ϕ is a semisimple element with 1 as an eigenvalue with multiplicity 1. Thus $C_L(\phi)$ is not contained in H , and so there exists $g \in L$ such that $H \cap H^g \cap L = \langle \phi \rangle \cong C_d$, that is, there exists a subdegree of length ℓ , which by Lemma 6.4 is not divisible by d . Thus it is sufficient to find a subdegree divisible by d . Since $\nu(\phi) = d - 1$, [9, Proposition 3.36] and [27, Theorem 4.2.2(j)] imply that $|\phi^G| > \frac{1}{2} \left(\frac{q}{q+1} \right)^{d-1} q^{d(d-1)} > \frac{1}{2} q^{(d-1)^2} > q^{(d-1)^2-1}$. On the other hand, $|\phi^G \cap H| < |H \cap L| < q^d \cdot d = q^{d+\log_q(d)} \leq q^{d+\log_2(d)}$. Hence for $d \geq 5$, $|\phi^G| > |\phi^G \cap H|^2$ and so by Lemma 2.6(ii), G has a subdegree divisible by d . For $d = 3$ and $\epsilon = +$ we have $q \geq 4$ and $|\phi^G| > \frac{1}{2}q^6$ while $|\phi^G \cap H| \leq (q^2 + q + 1) \cdot 2$. It follows that $|\phi^G| > |\phi^G \cap H|^2$ and so by Lemma 2.6(ii), G has a subdegree divisible by $d = 3$. For $\epsilon = -$, note that $q \geq 4$ and so $|\phi^G| > \frac{1}{2} \left(\frac{q}{q+1} \right)^2 q^6 > \frac{1}{2}q^5$. Now $|\phi^G \cap H| \leq (q^2 - q + 1)2$ and so for $q \geq 7$, $|\phi^G| > |\phi^G \cap H|^2$. For $q = 5$, we actually have $|\phi^G \cap H| \leq (q^2 - q + 1)2/3$ which is sufficient to show $|\phi^G| > |\phi^G \cap H|^2$. Thus for $q \geq 5$, Lemma 2.6

yields a subdegree divisible by d . For $L = \text{PSU}_3(4)$, a MAGMA [7] calculation shows that the subdegrees of L are 1, 13^9 and 39^{38} . Thus G is not $\frac{3}{2}$ -transitive in this case also.

Suppose now that $d = 2$ and q is odd. Then $H \cap L \cong D_{q+1}$. Suppose first that $(q+1)/2$ is odd. Then $|H|$ and $|\Omega|$ are even. Let $x \in H \cap L$ have order 2. Then $|x^G| = q(q+1)/2$ while $|x^G \cap H| = (q+1)/2$. It follows from Lemma 2.6(ii) that G has an even subdegree and so G is not $\frac{3}{2}$ -transitive. Suppose now that $(q+1)/2$ is even. Then H is the centraliser of an involution and for each $g \in G \setminus H$ we have $L \cap H \cap H^g = 1, C_2$ or C_2^2 . Now $H \cap L$ contains $(q+3)/2$ involutions and so each involution is contained in $(q+3)/2$ conjugates of H . The number of conjugates of H containing an involution of $H \cap L$ is at most $(q+3)^2/4$. Then since there are $q(q-1)/2$ conjugates of H and $q(q-1)/2 > (q+3)^2/4$ for $q \geq 11$, it follows that in these cases there exists $g \in G \setminus H$ such that $H \cap H^g \cap L = 1$, and hence a subdegree divisible by $q+1$. A MAGMA [7] calculation verifies the existence of such a subdegree for $q = 7$. Since $q+1$ does not divide $|\Omega| - 1$ it follows that G is not $\frac{3}{2}$ -transitive. \square

6.4. Aschbacher class \mathcal{C}_4 : Here H is the stabiliser in G of a tensor product structure $V = U \otimes W$ where $\dim(U) = a$, $\dim(W) = b$ and $d = ab$ with $a, b \geq 2$ and U and W are not isometric. Detailed descriptions of the subgroups are given in [34, §4.4]. We note that by [51, Lemma 3.7], if $g = (g_1, g_2) \in \text{GL}(U) \otimes \text{GL}(W)$ then $\nu(g) \geq \max\{a\nu(g_2), b\nu(g_1)\}$, with ν as in Definition 5.6. Moreover, if g_1 has order p and $g_2 = 1$ then $\nu(g) = b\nu(g_1)$ and similarly, if g_2 has order p and $g_1 = 1$ then $\nu(g) = a\nu(g_2)$.

By [34, Table 4.4A] the possibilities for G and H are as follows:

- (1) G with socle $\text{PSL}_d(q)$ and H of type $\text{GL}_a(q) \otimes \text{GL}_b(q)$ with $a < b$,
- (2) G a unitary group and H of type $U_a(q) \otimes U_b(q)$ with $a < b$,
- (3) G a symplectic group and H of type $\text{Sp}_a(q) \otimes O_b^\epsilon(q)$ with q odd and $b \geq 3$ and $a \geq 2$,
- (4) G of type O^+ and H of type $\text{Sp}_a(q) \otimes \text{Sp}_b(q)$ with $2 \leq a < b$,
- (5) G an orthogonal group and H of type $O_a^{\epsilon_1}(q) \otimes O_b^{\epsilon_2}(q)$ with q odd, $a, b \geq 3$ and $(a, \epsilon_1) \neq (b, \epsilon_2)$.

Suppose first that $L = \text{PSL}_d^\epsilon(q)$ and let $g = (1, x) \in H$ of order p such that $\nu(x) = 1$. Thus $\nu(g) = a < b$ and so if $h \in g^G \cap H$ then $h = (1, x_1)$ for some $x_1 \in \text{PSL}_b^\epsilon(q)$ of order p with $\nu(x_1) = 1$. Since $a < d/2$, Proposition 5.7 implies that $|g^G| > \frac{1}{2} \frac{q}{(q-\epsilon)(q+1)} q^{2a(d-a)} > q^{2a^2(b-1)-4}$ while by Lemma 5.9 $|g^G \cap H| = (q^b - (\epsilon^b))(q^{b-1} - (\epsilon^{b-1}))/ (q - \epsilon) < q^{2b-1}$. Since $b \geq 3$, it follows from Lemma 2.6(ii) that G has a subdegree divisible by p .

Next suppose that G is a symplectic group and H is of type $\text{Sp}_a(q) \otimes O_b^\epsilon(q)$ with q odd and $b \geq 3$. Let $g = (x, 1) \in H$ of order p with $\nu(x) = 1$. Then $\nu(g) = b \geq 3$ and by Proposition 5.7, $|g^G| > \frac{q}{4(q+1)} q^{b(ab-b)} > q^{b^2(a-1)-2}$. Since $O_b^\epsilon(q)$ does not contain any unipotent elements with $\nu(y) = 1$, if $a > b/2$ then $|g^G \cap H|$ is the number of transvections in $\text{Sp}_a(q)$. Thus by Lemma 5.9, $|g^G \cap H| < q^a$. Since $a \geq 2$, it follows from Lemma 2.6 that G has a p -subdegree. If $a \leq b/2$, then H has a weakly closed normal subgroup $H_0 \cong \text{PO}_b^\epsilon(q)$. By Proposition 5.2, H_0 contains a cyclic subgroup T such that for any Sylow p -subgroup S of H we have $H_0 \leq \langle T, S \rangle$. If $\epsilon = +$ then T centralises a 2-subspace of W and hence a $2a$ -subspace of V . Thus $N_H(T) < N_G(T)$. If $\epsilon = -$, then T acts irreducibly on W and preserves a direct sum decomposition of V into a totally singular subspaces of V each with dimension b . Now T is irreducible on each subspace in the decomposition and $|T|$ is at most $q^{b/2} + 1$. Thus T is centralised by an element of order $q^b - 1$ which also fixes this decomposition and is not in H . Thus by Lemma 2.2, G has a subdegree divisible by p .

Next suppose that G is of type O^+ and H is of type $\text{Sp}_a(q) \otimes \text{Sp}_b(q)$ with $a < b$. Let $g = (1, x) \in H$ of order p where $\nu(x) = 1$. Then $\nu(g) = a$ and $h \in g^G \cap H$ if and only if $h = (1, x_1)$ for some $x_1 \in x^{\text{Sp}_b(q)}$ with $\nu(x_1) = 1$. By Proposition 5.7, $|g^G| > \frac{1}{16} q^{a(d-a-1)} > q^{a^2(b-1)-a-4}$ while by Lemma 5.9, $|g^G \cap H| < q^b$. Since $b \geq 4$, Lemma 2.6(ii) implies that G has a subdegree divisible by p except when $(a, b) = (2, 4)$. In this case, [3, (15.1)] implies that H is conjugate under a triality automorphism to the stabiliser of a nondegenerate 3-space when q is odd, while when q is even H is conjugate to a subgroup of the stabiliser of a nonsingular 1-space. Hence we have already seen that there is a subdegree divisible by p .

Finally suppose that G is an orthogonal group and H is of type $O_a^{\epsilon_1}(q) \otimes O_b^{\epsilon_2}(q)$ with q odd, $a, b \geq 3$ and $(a, \epsilon_1) \neq (b, \epsilon_2)$. We may assume that $a \leq b$. Let $g = (1, x) \in H$ of order p where $\nu(x) = 2$. Then $\nu(g) = 2a$ and if $a < b$ then $h \in g^G \cap H$ if and only if $h = (1, x_1)$ for some $x_1 \in x^{O_b^{\epsilon_2}(q)}$ with $\nu(x_1) = 2$. By Proposition 5.7, $|g^G| > \frac{q}{8(q+1)} q^{2a(ab-2a-1)} > q^{2a(ab-2a-1)-3}$, while by [9, Proposition 3.22], $|g^G \cap H| < 2q^{2b-4} < q^{2b-3}$. Since $a, b \geq 3$, it follows from Lemma 2.6(ii) that G has a subdegree divisible by p . It remains to consider the case $a = b$ and $\epsilon_1 \neq \epsilon_2$. Note this implies that a is even. If $(x_1, x_2) \in g^G \cap H$ then $\nu(x_i) \in \{0, 2\}$. Thus $|g^G \cap H| < q^{4b-6}$. Again, as $a = b \geq 4$, Lemma 2.6(ii) yields a p -subdegree.

In all cases we have found a p -subdegree and so as p divides $|\Omega|$, G is not $\frac{3}{2}$ -transitive.

6.5. Aschbacher class \mathcal{C}_5 : Here H is the stabiliser of a subfield structure for a subfield $\text{GF}(q_0)$ where $q = q_0^r$ with r a prime. Descriptions of these groups can be found in [34, §4.5].

Suppose first that H is of the same Lie type as G and we initially exclude the cases where $L = \text{PSL}_2(q)$ and $\text{PSU}_3(q)$ with H of the form $\text{PSL}_2(3)$ and $\text{PSU}_3(2)$ respectively. Note also that if $d = 2$ then $q_0 \neq 2$. Then H has a weakly closed normal subgroup H_0 which is an insoluble classical group. By Proposition 5.2, there exists a cyclic subgroup T of order given by Table 4, with q replaced by q_0 such that given a Sylow p -subgroup S of H , $H_0 \leq \langle T, S \rangle$. Moreover, T is centralised by the subgroup \bar{T} of G given by Proposition 5.2. Hence $N_H(T) < N_G(T)$ and so by Lemma 2.2, G has a subdegree divisible by p .

Now let $L = \text{PSL}_2(q)$ and $q_0 = 3$. Let $x \in H$ be a transvection. Then $|x^G| \geq \frac{1}{2}(q^2 - 1)$ while $|x^G \cap H| \leq 8$. It follows from Lemma 2.6(ii) that when $q \geq 27$, the group G has a subdegree divisible by p . When $q = 9$, a Magma [7] calculation verifies that there is a subdegree of size 6. For $L = \text{PSU}_3(q)$, q even, and $H \cap L = \text{PSU}_3(2) \cong 3^2 : Q_8$, let $x \in H \cap L$ be an involution. Note also that r is odd [34, §4.5], so $q \geq 8$. Then $|x^G \cap H| = 9$ while by Proposition 5.7, $|x^G| > q^5/(2(q+1)^2) > 202$. Hence by Lemma 2.6(ii), G has a subdegree divisible by $p = 2$.

Next suppose that H is of a different Lie type to G . Then by [34, Table 4.5A] the possibilities are:

- $L = \text{PSU}_d(q)$ and H of type $O_d^\epsilon(q)$ with $\epsilon \in \{+, -, \circ\}$ and q odd.
- $L = \text{PSU}_d(q)$ and H of type $\text{Sp}_d(q)$.
- $L = \text{P}\Omega_d^+(q)$ and H of type $O_d^-(q^{1/2})$.

Then if q is odd and H is not of type $O_3(3)$ or $O_4^+(3)$, H contains a weakly closed normal subgroup H_0 which is a nonlinear classical group and by [43], $|G : H|$ is even. Hence by Lemma 2.4, G has a subdegree divisible by p .

If $L = \text{PSU}_3(3)$ and $H \cap L \cong \text{PSO}_3(3)$ we see from [22, p14] that H is not maximal in G . Similarly, for $L = \text{PSU}_4(3)$ and $H \cap L \cong \text{PSO}_4^+(3)$, [22, p52] shows that H is not maximal in G .

Next suppose that $L = \text{PSU}_d(q)$ for q even and $H = \text{PSp}_d(q)$ (see [34, Proposition 4.5.6]). By Proposition 5.2, H has a cyclic subgroup T of order $q^{d/2} + 1$ such that $H = \langle S, T \rangle$ for any Sylow 2-subgroup S of H . Now T is self-centralising in H but we claim that T is contained in a torus \bar{T} of order $q^d - 1$ in L . If $d/2$ is odd one can see this by viewing $T < \text{Sp}_2(q^{d/2}) < \text{GU}_2(q^{d/2}) < \text{GU}_d(q)$: there is a torus \bar{T} of order $q^d - 1$ in $\text{GU}_2(q^{d/2})$ containing T , and the $\text{Sp}_2(q^{d/2})$ is in $\text{Sp}_d(q) < \text{GU}_d(q)$; and if $d/2$ is even then if one takes an element in $\text{GU}_d(q)$ of order a primitive prime divisor² of $q^d - 1$, its centralizer in $\text{Sp}_d(q)$ is T and its centralizer in $\text{GU}_d(q)$ is \bar{T} . Thus in both cases, $N_H(T) < N_G(T)$ and so by Lemma 2.2, G has a subdegree divisible by p .

Finally suppose $L = \text{P}\Omega_d^+(q)$ for q even and $H = \text{P}\Omega_d^-(q^{1/2})$ (see [34, Proposition 4.5.10]). By Proposition 5.2, H has a cyclic subgroup T of order $q^{d/4} + 1$ such that $H = \langle S, T \rangle$ for any Sylow 2-subgroup S of H . Now a Singer cycle of order $(q^{1/2})^{d/2} + 1$ is $\Omega_2^-((q^{1/2})^{d/2})$. Moreover, $\Omega_d^+(q)$ has a \mathcal{C}_3 -subgroup $\Omega_2^+(q^{d/2})$, which is a torus of order $q^{d/2} - 1$, and this contains $\Omega_2^-((q^{1/2})^{d/2})$, which is the original Singer cycle. Since $d \geq 8$ it follows that T is self-centralising in $\Omega_d^-(q)$ but not in G . Thus $N_H(T) < N_G(T)$ and so by Lemma 2.2, G has a subdegree divisible by p .

In all cases we have found a p -subdegree and $|\Omega|$ is divisible by p . Hence G is not $\frac{3}{2}$ -transitive.

6.6. Aschbacher class \mathcal{C}_6 : Here H is the normaliser of a symplectic-type r -group. Descriptions can be found in [34, §4.6].

First we deal with the case where $d = 2$. Here $H \cap L = A_4$ or S_4 . Moreover, $q = p \geq 5$ and so p does not divide $|H|$. We note first that if $p = 5$ or 7 then G is 2-transitive and so we assume that $q \geq 11$. Let $x \in H$ have order 3 and note that 3 divides either $q - 1$ or $q + 1$. Then $|x^G| = q(q \pm 1)$ while $|x^G \cap H| = 8$. Hence by Lemma 2.6(ii), G has a subdegree divisible by 3. Let S be a Sylow 3-subgroup of H . Then $N_H(S) \leq S_3$ while $S_3 < D_{q-\epsilon} \leq N_G(S)$. Thus there exists $g \in G \setminus H$ such that $S \leq H \cap H^g$ and so G also has a subdegree not divisible by 3, showing that G is not $\frac{3}{2}$ -transitive.

For $d > 2$ and $L = \text{PSL}_d^\epsilon(q)$, there are two types of \mathcal{C}_6 subgroups: those of type $r^{1+2m}.\text{Sp}_{2m}(r)$ for r odd, $q \equiv \epsilon \pmod{r}$ and $d = r^m$, and those of type $(4 \circ 2^{1+2m}).\text{Sp}_{2m}(2)$ for $q = p \equiv \epsilon \pmod{4}$ and $d = 2^m$. In the first case, $q = p^f$ where f is the smallest integer such that $p^f \equiv \epsilon \pmod{r}$. By [34, Propositions 4.6.5], if H is of the first type then

$$H \cap L = \begin{cases} 3^2.Q_8 & d = 3 \text{ and } q \equiv \epsilon 4 \text{ or } \epsilon 7 \pmod{9} \\ r^{2m}.\text{Sp}_{2m}(r) & \text{otherwise} \end{cases}$$

²A primitive prime divisor of $q^d - 1$ is a prime divisor of $q^d - 1$ that does not divide $q^i - 1$ for any $i < d$.

If H is of type $(4 \circ 2^{1+2m}).\text{Sp}_{2m}(2)$, then by [34, Proposition 4.6.6],

$$H \cap L = \begin{cases} 2^4.A_6 & d = 4 \text{ and } p \equiv \epsilon 5 \pmod{8} \\ 2^{2m}.\text{Sp}_{2m}(2) & \text{otherwise} \end{cases}$$

Suppose first that H is of type $(4 \circ 2^{1+2m}).\text{Sp}_{2m}(2)$. Since $q = p$, p divides $|H|$ if and only if p divides $|H \cap L|$. Suppose then that p divides $|H \cap L|$ and $x \in H \cap L$ has order p . If $m \geq 4$, then [9, Lemma 6.3] implies that $\nu(x) \geq 4$. Then by Proposition 5.7, $|x^G| > \frac{q}{2(q-\epsilon)(q+1)}q^{8(d-4)} > q^{2^{m+3}-35}$. Now $|H \cap L|^2 < 2^{4m^2+6m}$ and so $|x^G| > |x^G \cap H|^2$. Thus by Lemma 2.6(ii), G has a p -subdegree. For $m = 3$, [9, Lemma 6.3] implies that $\nu(x) \geq 2$ and so $|x^G| > \frac{q}{2(q-\epsilon)(q+1)}q^{4(d-2)} > q^{2^5-11} = q^{21}$. Also $H \cap L = 2^6.\text{Sp}_6(2)$ and so for $q \geq 7$, we have $|x^G| > |x^G \cap H|^2$. Thus Lemma 2.6(ii) once again yields a p -subdegree. It remains to consider $(\epsilon, d, q) = (+, 8, 5)$ or $(-, 8, 3)$. When $q = 3$, GAP [25] calculations show that $H \cap L$ contains precisely 143360 elements x of order 3 with $\nu(x) = 5$. By Proposition 5.7, for such an element x we have $|x^L| > 3^{37} > |x^L \cap H|^2$ and so Lemma 2.6 implies the existence of a subdegree divisible by $p = 3$. When $q = 5$, GAP [25] calculations show that there is a regular suborbit and hence one divisible by $p = 5$. (In both the $p = 3$ and 5 cases we can construct $H \cap L$ using the algorithm outlined in [30, Section 9]. To find the existence of a regular suborbit we simply choose random elements $g \in L$ until we find one with $H \cap H^g \cap L = 1$.) Finally, if $m = 2$ note that $H \cap L \leq 2^4.\text{Sp}_4(2)$, and so if $|H \cap L|$ is divisible by p then either $\epsilon = +$ and $p = 5$, or $\epsilon = -$ and $p = 3$. For $L = \text{PSL}_4(5)$, we have $H \cap L = 2^4.A_6$ and using MAGMA [7] we see that the subdegrees for L are

$$1, 16^2, 30, 80, 96^2, 120, 160, 240^6, 320, 360, 480^6, 960^{13}, 1152^{10}, 1440^{10}, 1920^{21}, 2880^{28}, 5760^{190}$$

Thus L , and hence G , has a subdegree divisible by $p = 5$. For $L = \text{PSU}_4(3)$, a MAGMA [7] calculation shows that the subdegrees for L are 30, 96, 120 and 320. Thus we have found a subdegree divisible by 3.

To show that G is not $\frac{3}{2}$ -transitive, note that H has a normal 2-subgroup and so Lemma 2.1(iv) implies that G has an even subdegree. By [43], $|\Omega|$ is even and hence G is not $\frac{3}{2}$ -transitive.

Next suppose that H is of type $r^{1+2m}.\text{Sp}_{2m}(r)$. Now $q = p^f$ where f is the smallest integer such that $p^f \equiv \epsilon \pmod{r}$. Hence f divides $r - 1$. It follows that if p divides $|H|$ then p divides $|H \cap L|$. Let $x \in H \cap L$ have order p . If $(r, m) = (3, 1)$ then the divisors of $|H \cap L|$ imply that $p = 2$. Hence $(\epsilon, q) = (+, 4)$ or $(-, 2)$. The latter is not possible as we have excluded $\text{PSU}_3(2)$. For $L = \text{PSL}_3(4)$, a MAGMA [7] calculation shows that the subdegrees are $\{1, 72^3, 18^3, 9\}$ which include subdegrees divisible by 3. Thus we may assume that $d \geq 5$ and so by [9, Lemma 6.3], $\nu(x) \geq 2$. Hence by Proposition 5.7, $|x^G| > q^{4.r^m-12}$. Now $H \cap L \leq r^{2m}.\text{Sp}_{2m}(r)$ and so $|x^G \cap H|^2 < r^{4m^2+6m}$. Now $r \leq q + 1$ so $|x^G \cap H|^2 < q^{8m^2+12m}$. Thus for $r \geq 11$, Lemma 2.6(ii) yields a p -subdegree. For $r = 7$, note that $q \geq 8$ and so $|x^G| > |x^G \cap H|^2$ for all values of m . Hence Lemma 2.6(ii) then yields a p -subdegree. For $r = 5$ we have $q \geq 4$ and we see that $|x^G| > |x^G \cap H|^2$ except when $(q, m) = (4, 1)$. In this case $H \cap L = 5^2.\text{Sp}_2(5)$, which contains a unique conjugacy class of involutions. Moreover, such involutions have centraliser $\text{Sp}_2(5)$ in $H \cap L$. Hence $|x^G \cap H|^2 = 5^4 < |x^G|$ and Lemma 2.6(ii) again yields a p -subdegree. Finally, suppose that $r = 3$ and note that $d \geq 5$ implies that $m \geq 2$. If $q \geq 4$ then $|x^G| > 4^{4.3^m-12} > 3^{4m^2+6m} > |x^G \cap H|^2$ and so Lemma 2.6(ii) yields a p -subdegree. If $q = 2$ then Lemma 2.6(ii) yields a p -subdegree except in the case where $m = 2$. In this case $H \cap L = 3^4.\text{Sp}_4(3)$ and $L = \text{PSU}_9(2)$. Then $\text{Sp}_4(3)$ contains a central element of order 2 and 90 elements of order 2 whose -1 eigenspace has dimension 2. Hence $|x^G \cap H| < 3^4 + 90.3^2$. Thus $|x^G \cap H|^2 < 2^{24} < |x^G|$ and so by Lemma 2.6(ii), G has a subdegree divisible by $p = 2$.

We now consider $\frac{3}{2}$ -transitivity. For q even we have just shown that we can always find an even subdegree. Since $|\Omega|$ is even this implies that G is not $\frac{3}{2}$ -transitive. Hence we may assume that q is odd. Suppose first that $d = r = 3$ and $q \equiv \epsilon 4, \epsilon 7 \pmod{9}$. Here $H \cap L = 3^2.Q_8$. Let $x \in H \cap L$ be an involution. Then $|x^G \cap H| = 9$. There is a unique conjugacy class of involutions in $\text{PSL}_3^\epsilon(q)$ and $|x^L| = q^2(q^2 + \epsilon q + 1)$. Since $L \neq \text{PSU}_3(2)$, it follows that $|x^G| > |x^G \cap H|^2$ and so Lemma 2.6(ii) implies that there is an even subdegree. By [43], $|\Omega|$ is even and so G is not $\frac{3}{2}$ -transitive.

If $d > 3$, or $d = 3$ and $q \equiv \epsilon \pmod{9}$, note that by Lemma 2.1(iv), G has a subdegree divisible by r . The order of L is divisible by $(q - \epsilon)^{d-2}$ and hence divisible by r^{d-2} . Moreover, since $q \equiv \epsilon \pmod{r}$, $(q^r - \epsilon)/(q - \epsilon)$ is also divisible by r . Hence for each term $q^{ir} - 1$ with i even we get an extra r dividing $|L|$ and for each term $q^{ir} - \epsilon$ with i odd we also get an extra r dividing $|L|$. Since $d = r^m$ it follows that $|L|$ is divisible by $r^{r^m-2}r^{r^{m-1}} = r^{r^m+r^{m-1}-2}$. On the other hand the largest power of r dividing $|H \cap L|$ is r^{m^2+2m} . Hence $|\Omega|$ is divisible by r and so G is not $\frac{3}{2}$ -transitive.

Next suppose that $L = \text{PSp}_d(q)$ where $q = p$ and $d = 2^m \geq 4$ and H is of type $2_-^{1+2m}.O_{2m}^-(2)$. Suppose that p divides $|H|$. Then p divides $|H \cap L|$ and so let $x \in H \cap L$ have order p . We first suppose

that $m \geq 4$ and so by [9, Lemma 6.3] we have that $\nu(x) \geq 4$. Then by Proposition 5.7, it follows that $|x^G| > \frac{q}{4(q+1)}q^{4(d-4)} > q^{2^{m+2}-18}$. By [34, Proposition 4.6.9], $H \cap L \cong 2^{2m}.\Omega_{2m}^-(2).c$ where $c = 1$ if $p \equiv \pm 3 \pmod{8}$ and $c = 2$ if $p \equiv \pm 1 \pmod{8}$. Hence

$$|H \cap L| = 2^{c-1} \cdot 2^{2m} \cdot 2^{m(m-1)}(2^m + 1) \prod_{i=1}^{m-1} (2^{2i} - 1) < 2^{2m^2+m+2}$$

Thus $|H \cap L|^2 < q^{4(2m^2+m+2)/3} < |x^L|$ for $m \geq 5$. For $m = 4$, explicitly calculating $|H \cap L|$ also yields $|H \cap L|^2 < |x^G|$. Hence there is a p -subdegree by Lemma 2.6(ii).

When $m = 3$ we have that $|H \cap L|$ divides $2^{13} \cdot 3^4 \cdot 5$ and so $p = 3$ or 5 . Thus $H \cap L = 2^6.\Omega_6^-(2)$. When $p = 3$, a MAGMA [7] calculation shows that $H \cap L$ contains 5120 elements x of order 3 such that $\nu(x) = 5$. Thus by Proposition 5.7, $|x^G| > \frac{1}{16}3^{21}$ and so by Lemma 2.6(ii), G has a subdegree divisible by 3. For $p = 5$, a MAGMA [7] calculation shows that $H \cap L$ contains 82944 elements of order 5 and for each such element x , $\nu(x) = 6$. Thus by Proposition 5.7, $|x^G| > \frac{1}{24}5^{25}$ and so by Lemma 2.6(ii), G has a subdegree divisible by 5.

For $m = 2$, $|H \cap L|$ divides $2^7 \cdot 3 \cdot 5$ and so $p = 3$ or 5 . Hence $H \cap L = 2^4.\Omega_4^-(2)$. When $p = 5$ a MAGMA [7] calculation shows that the subdegrees for L are

$$1, 10, 40, 80, 120, 160^2, 192^2, 240, 320, 480, 960^3$$

of which many are divisible by $p = 5$. When $p = 3$ we see that the subdegrees of L are 1, 16 and 10, none of which are divisible by 3; this is an example in line 6 of Table 3 of Theorem 5.1(A).

Finally suppose that $L = \text{P}\Omega_d^+(q)$ where $q = p$ and H is of type $2_+^{1+2m}.\text{O}_{2m}^+(2)$. Suppose that p divides $|H|$. Then p divides $|H \cap L|$ and so let $x \in H \cap L$ have order p . By [3, p512], if $m = 3$ such subgroups are conjugate under a triality automorphism to the stabiliser of a 1-space decomposition and so this case has already been dealt with in Section 6.2. Thus we may assume that $m \geq 4$. Then by [9, Lemma 6.3] we have that $\nu(x) \geq 4$. Then by Proposition 5.7 it follows that $|x^G| > \frac{q}{8(q+1)}q^{4(d-4-1)} > q^{2^{m+2}-23}$. By [34, Proposition 4.6.9], $H \cap L \cong 2^{2m}.\Omega_{2m}^+(2).c$ where $c = 1$ if $p \equiv \pm 3 \pmod{8}$ and $c = 2$ if $p \equiv \pm 1 \pmod{8}$. Thus $|H \cap L|^2 \leq 2^{4m^2+4m+2} < q^{4(2m^2+m+1)/3} < |x^G|$ for $m \geq 5$. For $m = 4$ and $q \geq 5$, explicitly calculating $|H \cap L|$ also yields $|H \cap L|^2 < |x^G|$. Thus we have found p -subdegrees except in the case where $m = 4$ and $q = p = 3$. Since $\Omega_8^+(2)$ contains 365120 elements of order 3, $H \cap L$ contains at most 2^8 times this number. Thus $|x^G \cap H|^2 < |x^G|$ and we have once again found a p -subdegree.

To show that G is not $\frac{3}{2}$ -transitive when $L = \text{PSp}_d(q)$ or $\text{P}\Omega_d^+(q)$, note that Lemma 2.1(iv) implies that G has an even subdegree and by [43], $|\Omega|$ is even.

6.7. Aschbacher class \mathcal{C}_7 : Here H is the stabiliser of a tensor product decomposition $V = U_1 \otimes U_2 \otimes \cdots \otimes U_t$ where each U_i has dimension m and $d = m^t$ with $t \geq 2$. Descriptions of the groups can be found in [34, §4.7].

Suppose first that $L = \text{PSL}_d(q)$ or $\text{P}\Omega_d^\epsilon(q)$ with the additional assumption that if L is an orthogonal group then each U_i is also orthogonal. Then $|H \cap L| \leq |\text{GL}_m(q)|^t t! < q^{tm^2} t! < q^{3m^t}$. By [34, Tables 3.5A and 3.5E], $m \geq 3$ and when G is orthogonal q is odd. Let $x = ([J_3, I_{m-3}], I_m, \dots, I_m) \in H \cap L$, where

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then $\nu(x) = 2m^{t-1}$ and so by Proposition 5.7, $|x^G| > \frac{q}{q^t}(q^{m^t(2m^{t-1}-1)/2}) = q^{m^t(2m^{t-1}-1)/2-3}$. Now $6m^t \leq m^t(2m^{t-1}-1)/2-3$ if and only if $6 < (2m^{t-1}-1)/2$. Thus for $t \geq 3$ or $t = 2$ and $m \geq 7$ we have that $|x^G| > |H \cap L|^2$ and so by Lemma 2.6 we have a subdegree divisible by p . For $t = 2$ we in fact have that $|H \cap L| < 2q^{2m^2} < q^{2m^2+1}$ and $|x^G| > q^{m^2(2m-1)/2-3}$ and so $|x^G| > |H \cap L|^2$ for $m \geq 5$. For $m = 3$ we have $|x^G| > q^{39/2}$ and for $m = 4$ we have $|x^G| > q^{53}$. Using $|O_3(q)| < q^4$ and $|O_4^\epsilon(q)| < q^7$ we are then able to show that in the remaining orthogonal cases we also have $|x^G| > |H \cap L|^2$. For $L = \text{PSL}_{16}(q)$ note that by Proposition 5.7 we actually have $|x^G| > q^{126}$ and this yields $|x^G| > |H \cap L|^2$. Finally, for $L = \text{PSL}_9(q)$, consider the element $x = ([J_2, I_1], I_3) \in H \cap L$. Then $\nu(x) = 3$ and by Proposition 5.7 we have $|x^G| > q^{34}$. Now if $y \in x^G \cap H$ then either $y = (g_1, g_2)$ where $g_i \in \text{GL}_3(q)$ and is a transvection or the identity, or $y = (g, g^{-1})\sigma$ where $g \in \text{GL}_3(q)$ and σ interchanges the two factors of the tensor decomposition. Hence by Lemma 5.9 we have $|x^G \cap H| < (q^5)^2 + q^9 < 2q^{10} \leq q^{11}$. Thus $|x^G| > |x^L \cap H|^2$ and so by Lemma 2.6 there exists a subdegree divisible by p .

Next suppose that $L = \text{PSp}_d(q)$ or $\text{P}\Omega_d^+(q)$ with each U_i a symplectic space. Then $|H \cap L| \leq |\text{Sp}_m(q)|^t t! < q^{(m^2+m)t/2} t! < q^{2m^t}$. Let $x = ([J_2, I_{m-2}], I_m, \dots, I_m) \in H \cap L$ have order p . Now

$\nu(x) = m^{t-1}$ and so by Proposition 5.7, $|x^G| > \frac{1}{q^2} q^{m^{t-1}(m^t - m^{t-1} - 1)} = q^{m^{t-1}(m^t - m^{t-1} - 1) - 2}$. Now $4m^t \leq m^{t-1}(m^t - m^{t-1} - 1) - 2$ if $t \geq 3$ and $m \geq 3$, or $t = 2$ and $m \geq 6$. Hence for these values of m and t we have $|x^G| > |H \cap L|^2$ and so by Lemma 2.6 there is a subdegree divisible by p . By [34, Table 3.5C], if $t = 2$ then $L = \text{P}\Omega_d^+(q)$. Since $d \geq 8$ in this case it follows that $m \neq 2$. When $(m, t) = (4, 2)$ we in fact have that $|H \cap L| < q^{21}$ while $|x^G| > q^{42}$ and so we can again use Lemma 2.6 to find a subdegree divisible by p . We are left to consider the case where $m = 2$ and $t = 3$. By [34, Tables 3.5C and 3.5E] we have that $L = \text{PSp}_8(q)$ when q is odd and $\text{P}\Omega_8^+(q)$ when q is even. However, when q is even [3] implies that H is not maximal in G . Hence q is odd. Then Proposition 5.7 implies that $|x^G| > q^{30}$ while $|H \cap L| < 6q^9 < q^{11}$. Hence $|x^L| > |H \cap L|^2$ and Lemma 2.6 implies the existence of a subdegree divisible by p .

This leaves us to consider the case where $L = \text{PSU}_d(q)$. By [34, Table 3.5B] we have $m \geq 3$. Let $x = ([J_3, J_1^{m-3}], I_m, \dots, I_m) \in H \cap L$. Then $\nu(x) = 2m^{t-1}$ and so by Proposition 5.7, $|x^G| > q^{4m^{t-1}(m^t - 2m^{t-1}) - 4} = q^{4m^{2t-2}(m-2) - 4}$. Moreover, $|H \cap L| \leq |\text{GU}_m(q)|^{t!} < q^{(m^2+m)t+t \log_2 t} < q^{4m^t}$. Now $8m^t \leq 4m^{2t-2}(m-2) - 4$ if and only if $8 < 4m^{t-2}(m-2)$. Thus for $t \geq 3$ or $t = 2$ and $m \geq 5$ we have that $|x^L| > |H \cap L|^2$. For $t = 2$ we in fact have that $|H \cap L| < 2q^{2(m^2+m)} < q^{2(m^2+m)+1}$ and $|x^G| > q^{2m^3-4}$ and hence $|x^G| > |H \cap L|^2$ for $m \geq 3$. Hence Lemma 2.6 implies that there is a subdegree divisible by p .

In all cases we have found a subdegree divisible by p and so as p divides $|\Omega|$ (Lemma 2.3) it follows that G is not $\frac{3}{2}$ -transitive.

6.8. Aschbacher class C_8 : Here H is a classical group on V . Descriptions can be found in [34, §4.8]. If $L \neq \text{PSL}_d(q)$ then the only cases which occur are for $L = \text{PSp}_d(q)$ (q even) with H an orthogonal group. When $q = 2$ and $d \geq 6$ the two actions (of $\text{Sp}_d(2)$ on cosets of $O_d^\pm(2)$) are 2-transitive, as in Theorem 5.1(A)(ii). For $(d, q) = (4, 2)$, then the action of $\text{Sp}_4(2)'$ on the cosets of $O_4^\pm(2) \cap \text{Sp}_4(2)'$ is also 2-transitive and we have lines 1 and 2 of Table 3 of Theorem 5.1(A). For $q > 2$, from the proof of [41, Prop. 1] there is a unique suborbit of length $(q^d - \epsilon)(q^{d-1} + \epsilon)$ and $(q-2)/2$ of length $q^{d-1}(q^d - \epsilon)$. Hence there is a p -subdegree and G is not $\frac{3}{2}$ -transitive.

For $L = \text{PSL}_d(q)$ first we deal with the cases where H is soluble. These are $\text{SU}_3(2) \leq \text{SL}_3(4)$, $\text{SO}_3(3) \leq \text{SL}_3(3)$, $\text{SO}_4^+(3).2 \leq \text{SL}_4(3)$. It is not difficult to find the subdegrees directly by computer for L in these cases:

G	H	Subdegrees for L
$\text{SL}_3(4)$	$\text{SU}_3(2)$	1, 9, 18^3 , 72^3
$\text{SL}_3(3)$	$\text{SO}_3(3)$	1, 3, 4^2 , 6, 12^8 , 24^5
$\text{SL}_4(3)$	$\text{SO}_4^+(3).2$	1, 9, 18^2 , 24^2 , 32, 36^3 , 48^2 , 72^5 , 144^9 , 192, 288^{17} , 576^6

We see that in each example above, G has a subdegree divisible by p , and G is not $\frac{3}{2}$ -transitive. Moreover, any overgroup will also have an even subdegree and a subdegree divisible by p , so will not be $\frac{3}{2}$ -transitive.

From now on we may assume that H is insoluble. Let S be a Sylow p -subgroup of G . By Proposition 5.2, we can find a cyclic subgroup T of H which is either irreducible on V , or acts irreducibly on a hyperplane or codimension 2 subspace and trivially on the perp, such that $\langle S, T \rangle$ contains a weakly closed normal subgroup of H . Moreover, we easily see that $N_H(T) < N_G(T)$ and so Lemma 2.2 implies that there is a subdegree divisible by p . Since $|\Omega|$ is divisible by p , it follows that G is not $\frac{3}{2}$ -transitive.

6.9. C_9 groups. We use the following theorem of Burness, Guralnick and Saxl [10].

Theorem 6.6. *Let G be an almost simple classical group with socle L and let $H \in C_9$. Moreover, we suppose that L is not isomorphic to an alternating group and the action of G on the set of right cosets of H is not permutation isomorphic to a classical group acting on the set of right cosets of a C_i -subgroup with $1 \leq i \leq 8$. Then either the action of G on the set of cosets of H has a regular suborbit or $(L, H \cap L)$ is given by Table 9.*

We note that since G is not a Frobenius group, the existence of a regular suborbit implies that G is not $\frac{3}{2}$ -transitive. Moreover, if p divides $|H|$ then the regular suborbit has length divisible by p .

The C_9 -subgroups excluded by hypothesis from Theorem 6.6 are listed in [10, Table 2]. When the action of G is isomorphic to a classical group with a C_i -action for $i \leq 8$, the classical group has the same characteristic as G , so $\frac{3}{2}$ -transitivity and the existence of a subdegree divisible by p has already been determined in the previous sections. When L is isomorphic to an alternating group, since Section 4 has already considered when such groups can be $\frac{3}{2}$ -transitive, it remains to check for p -subdegrees. The actions under consideration here are when $L = \text{PSL}_2(9)$ of degree 6 on the set of cosets of $N_G(A_5)$, and

$L = \text{PSL}_4(2)$ of degree 8 on the cosets of $N_G(A_7)$. Both groups are 2-transitive and the unique nontrivial subdegree is not divisible by p . These provide the examples in lines 4 and 5 of Table 3 in Theorem 5.1(A).

TABLE 9. \mathcal{C}_9 actions without a regular suborbit

L	$H \cap L$	Subdegrees of L
$\Omega_7(q)$	$G_2(q)$, $q \equiv \epsilon \pmod{4}$, $\epsilon = \pm 1$	$q^6 - 1$, $q^3(q^3 + \epsilon)$, $(q - 4 - \epsilon)/4$ times $q^3(q^3 + 1)$, $(q - 2 + \epsilon)/4$ times $q^3(q^3 - 1)$
$\Omega_7(q)$	$G_2(q)$, q even	$q^6 - 1$, $q/2$ times $q^3(q^3 - 1)$, $(q - 2)/2$ times $q^3(q^3 + 1)$
$\text{PSp}_4(q)$, q even	$Sz(q)$	$(q - 1)(q^2 + 1)$, 2 times $\frac{1}{2}q(q - 1)(q^2 + 1)$, $(q - 2)/2$ times $q^2(q^2 + 1)$, $(q \pm \sqrt{2q})/4$ times $q^2(q - 1)(q \pm \sqrt{2q} + 1)$
$\text{PSL}_3(4)$	A_6	1, 10, 45
$\text{PSL}_2(19)$	A_5	1, 6, 20, 30
$\text{PSL}_2(11)$	A_5	1, 10
$\text{PSU}_6(2)$	$\text{PSU}_4(3).2$	1, 567, 840
	M_{22}	1, 77, 231, 1155, 1232, 2640, 6160, 9240
$\text{PSU}_4(3)$	$\text{PSL}_3(4)$	1, 56, 105
	A_7	1, 105, 140, 210 ² , 630
$\text{PSU}_3(5)$	$A_6.2$	1, 12, 90, 72
	A_7	1, 7, 42
	$\text{PSL}_3(2)$	1, 14 ² , 21, 28, 56 ³ , 84 ⁴ , 168
$\text{PSU}_3(3)$	$\text{PSL}_3(2)$	1, 7 ² , 21
$\text{Sp}_8(2)$	S_{10}	1, 210, 1575, 5600, 5670
$\text{Sp}_6(2)$	$\text{PSU}_3(3).2$	1, 56, 63
$\Omega_{14}^+(2)$	A_{16}	subdegrees include 130767436800, 290594304000, 435891456000, 653837184000, 871782912000, 1307674368000, 2615348736000, 3487131648000, 5230697472000, 10461394944000
$\Omega_{12}^-(2)$	A_{13}	subdegrees include 3603600, 14414400, 16216200, 21621600, 28828800, 32432400, 43243200, 48648600, 64864800, 86486400, 97297200, 129729600, 194594400, 259459200, 389188800, 518918400, 778377600, 1556755200
$\Omega_{10}^-(2)$	A_{12}	1, 462, 25202, 5775, 30800, 62370
$\text{P}\Omega_8^+(3)$	$\Omega_8^+(2)$	1, 960 ³ , 3150, 22400
$\Omega_8^+(2)$	A_9	1, 84, 315, 560
$\Omega_7(3)$	S_9	1, 126, 315, 560, 1680, 2520 ² , 4320, 5670, 7560
	$\text{Sp}_6(2)$	1, 288, 630, 2240

It remains to consider the actions listed in Table 9. The subdegrees for the infinite families are given in [41, Proposition 2] and [37, Theorem A], and we see that there are subdegrees divisible by p and that G is not $\frac{3}{2}$ -transitive. For the remaining cases, the subdegrees were calculated using MAGMA and are given in Table 9. The table shows that there are p -subdegrees in all cases where p divides $|H|$ except for $L = \text{PSU}_3(5)$ acting on A_7 , which is in line 3 of Table 3 of Theorem 5.1(A). Moreover, none of the groups are $\frac{3}{2}$ -transitive.

6.10. Graph automorphisms. When $L = \text{PSL}_d(q)$, $\text{Sp}_4(q)$ (q even) or $\text{P}\Omega_8^+(q)$ and G contains a graph automorphism (a triality in the latter case), there are extra maximal subgroups which we need to consider, and we do so in this section.

6.10.1. Linear groups. When $L = \text{PSL}_d(q)$ and G contains a graph automorphism there are two extra classes of subgroups that we need to consider.

First let Ω be the set of all pairs $\{U, W\}$ of complementary subspaces of V with $\dim(U) = m < d/2$. Let $x = \{\langle v_1, \dots, v_m \rangle, \langle v_{m+1}, \dots, v_d \rangle\} \in \Omega$ and $H = G_x$. Then $H \cap \text{P}\Gamma\text{L}_d(q)$ is the stabiliser of the decomposition $\langle v_1, \dots, v_m \rangle \oplus \langle v_{m+1}, \dots, v_d \rangle$. Now $y = \{\langle v_{m+1}, \dots, v_{2m} \rangle, \langle v_1, \dots, v_m, v_{2m+1}, \dots, v_d \rangle\} \in \Omega$ and $H_y \cap \text{P}\Gamma\text{L}_d(q)$ is the stabiliser of the decomposition $\langle v_1, \dots, v_m \rangle \oplus \langle v_{m+1}, \dots, v_{2m} \rangle \oplus \langle v_{2m+1}, \dots, v_d \rangle$. Hence $|H : H_y|$ is divisible by p and as p divides $|\Omega|$, it follows that G is not $\frac{3}{2}$ -transitive.

Finally, let $m < d/2$ and let Ω be the set of pairs of subspaces (U, W) of complementary dimensions with $\dim(U) = m$ and $U < W$. By Lemma 5.5, there is a unique subdegree which is a power of p . Since G is not 2-transitive, it is not $\frac{3}{2}$ -transitive.

6.10.2. 4-dimensional symplectic groups. When q is even, $\mathrm{Sp}_4(q)$ has a graph automorphism which interchanges totally isotropic 1-spaces and totally isotropic 2-spaces. Since $\mathrm{Sp}_4(2)' \cong A_6$ we have already checked for $\frac{3}{2}$ -transitivity in this case. It is straightforward to check that for all primitive groups with socle $\mathrm{Sp}_4(2)'$ the only ones with no even subdegrees are those of degrees 6 and 10, and these correspond to H being an orthogonal group, that is, we have the examples in lines 1 and 2 of Table 3 in Theorem 5.1(A). From now on we assume that $q > 2$. Aschbacher [3] gives three further classes of maximal subgroups of a group G with socle $L = \mathrm{PSp}_4(q)$ for $q > 2$ even, and containing a graph automorphism. These are as follows:

- (1) C'_1 : stabilisers of pairs $\{U, W\}$ of subspaces of V such that U is a totally isotropic 1-space and W is a totally isotropic 2-space containing U .
- (2) $N_G(X)$ where X is the stabiliser in $O_4^+(q)$ of a nondegenerate 2-space; here $N_L(X) = D_{2(q\pm 1)} \text{ wr } S_2$.
- (3) $N_G(C_{q^2+1})$: here $H \cap L = C_{q^2+1} \rtimes C_4$.

If $H \in C'_1$, then H has a normal 2-subgroup and so by Lemma 2.1(iv), G has an even subdegree. However, $|\Omega|$ is odd so to show that G is not $\frac{3}{2}$ -transitive we find two distinct subdegrees as follows. Let $\{e_1, e_2, f_1, f_2\}$ be a symplectic basis for V and suppose that H is the stabiliser of $x = \{\langle e_1 \rangle, \langle e_1, e_2 \rangle\}$. Then

$$L_x^{\langle e_1, e_2 \rangle} = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ \mu & \lambda_2 \end{pmatrix} \mid \lambda_1, \lambda_2 \in \mathrm{GF}(q) \setminus \{0\}, \mu \in \mathrm{GF}(q) \right\}.$$

Let $y = \{\langle e_2 \rangle, \langle e_1, e_2 \rangle\}$. Then $|G_x : G_{xy}| = 2q$ as G_{xy} can no longer interchange $\langle e_1 \rangle$ and $\langle e_1, e_2 \rangle$. Next let $z = \{\langle e_1 + e_2 \rangle, \langle e_1 + e_2, f_1 + f_2 \rangle\}$. If $g \in L_{xz}$ maps e_1 to $\lambda_1 e_1$ and e_2 to $\mu e_1 + \lambda_2 e_2$ we must have that $\lambda_2 = \lambda_1 + \mu$. Thus $q-1$ divides $|L_x : L_{xz}|$ and hence also $|G_x : G_{xz}|$. Since $q > 2$ it follows that G is not $\frac{3}{2}$ -transitive.

Next suppose that $H = N_G(X)$ where X is the stabiliser in $O_4^+(q)$ of a nondegenerate 2-space. Now $O_4^+(q)$ contains an involution $y = [J_2^2]$ where the two Jordan blocks are hyperbolic 2-spaces and also an involution $z = [J_2]$ where the two Jordan blocks are elliptic 2-spaces. Both y and z are of type c_2 in the notation of [4]. Hence by Proposition 5.7 we can always choose an involution $x \in H \cap L$ such that $|x^G| > \frac{1}{2}q^6$. Now $H \cap L = D_{2(q\pm 1)} \text{ wr } S_2$ which contains at most $(q+1)^2 + 2(q+1) = (q+1)(q+3)$ involutions. Since $q > 2$, it follows from Lemma 2.6(ii) that G has an even subdegree. Since $|\Omega|$ is even, G is not $\frac{3}{2}$ -transitive.

Finally suppose that $H = N_G(X)$ where $X \cong C_{q^2+1}$. Here $H \cap L$ is contained in the extension field subgroup $\mathrm{Sp}_2(q^2).2$ and $H \cap L = C_{q^2+1} \rtimes C_4$. Let $x \in H \cap L$ be an involution. Then x is an involution of $\mathrm{Sp}_2(q^2)$. Letting $\{e_1, f_1\}$ be a symplectic basis for V over $\mathrm{GF}(q^2)$ we may assume that x interchanges e_1 and f_1 . If $B : V \times V \rightarrow \mathrm{GF}(q^2)$ is the $\mathrm{GF}(q^2)$ -alternating form preserved by $H \cap L$ then we can take $\overline{B} = \mathrm{Tr}_{q^2 \rightarrow q} \circ B : V \times V \rightarrow \mathrm{GF}(q)$ to be the $\mathrm{GF}(q)$ -alternating form. Given $\mu \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$ we have $\mathrm{Tr}(\mu^2) \neq 0$. Thus as an element of $\mathrm{Sp}_4(q)$ we have that $\nu(x) = 2$ and x maps μe_1 to μf_1 with $\overline{B}(\mu e_1, \mu f_1) = \mathrm{Tr}(\mu^2) \neq 0$. Hence by [4, (7.6)], x is of type c_2 . Thus Proposition 5.7 implies that $|x^G| > \frac{1}{2}q^6$. Since $|x^G \cap L| = q^2 + 1$, Lemma 2.6(ii) implies that G has an even subdegree. Since $|\Omega|$ is even, G is not $\frac{3}{2}$ -transitive.

6.10.3. 8-dimensional orthogonal groups. The group $L = \mathrm{P}\Omega_8^+(q)$ has a graph automorphism of order three which permutes the set of totally singular 1-spaces and the two classes of maximal totally singular 4-spaces. Any automorphism of G that induces a permutation of order three on the corresponding three classes of subgroups is called a *triality automorphism*. If G is an almost simple group with socle L and contains such a triality automorphism then there are several extra families of maximal subgroups that we need to consider. These are given in [35] and are as follows. We let $d = (2, q-1)$.

- (1) $G_{\{U, X, W\}}$ where U is a totally singular 1-space, X and W are totally singular 4-spaces with $\dim(X \cap W) = 3$, and $U < X \cap W$.
- (2) $N_G(G_2(q))$.
- (3) $N_G(N_1)$, where N_1 is the intersection of the stabiliser in L of an anisotropic 2-space with the normaliser in L of an irreducible $\mathrm{SU}_4(q)$. The preimage of N_1 in $\Omega_8^+(q)$ is $(\frac{1}{d}C_{q+1} \times \frac{1}{d}\mathrm{GU}_3(q)).2^d$.
- (4) $N_G(N_2)$, where N_2 is the intersection of the stabiliser in L of a hyperbolic line $\langle e_1, f_1 \rangle$ and the stabiliser in L of a decomposition of V into two totally singular 4-spaces containing e_1 and f_1 respectively. Moreover, $q \neq 2$. The preimage of N_2 in $\Omega_8^+(q)$ is $(\frac{1}{d}C_{q-1} \times \frac{1}{d}\mathrm{GL}_3(q)).2^d$.
- (5) $N_G(N_3)$ where $N_3 = (D_{\frac{2}{d}(q^2+1)} \times D_{\frac{2}{d}(q^2+1)}).2^2$.

- (6) $N_G(N_4)$, where $N_4 = [2^9] \rtimes \text{PSL}_3(2)$ and $q = p \geq 3$. (We use $[2^9]$ to denote an unspecified group of order 2^9 .) When $q \equiv \pm 3 \pmod{8}$ these have odd index.

If H is as in case (1) then by Lemma 5.5, G has a unique subdegree which is a power of p . Hence G is not $\frac{3}{2}$ -transitive.

If H is as in case (2) then $H \cap L = G_2(q) < \text{P}\Omega_7(q) < L$. It was seen in Table 9, that $\text{P}\Omega_7(q)$ in its action on cosets of $G_2(q)$ has a subdegree divisible by p and so by Lemma 2.1(ii), so does L and hence G also. Hence G is not $\frac{3}{2}$ -transitive (since p divides $|\Omega|$).

If H is as in case (3), note that $N_1 = L \cap H < R < L$, where R is a 4-dimensional unitary group whose matrices have entries from $\text{GF}(q^2)$. Moreover, the action of R on the set of cosets of N_1 in R is the primitive action of R on nondegenerate 1-spaces over $\text{GF}(q^2)$. We have already seen in Section 6.1 that this action of R has a subdegree divisible by p and so by Lemma 2.1(ii), L , and hence G , has a subdegree divisible by p . Since p divides $|\Omega|$ it follows that G is not $\frac{3}{2}$ -transitive.

For H in case (4), note that $N_2 = L \cap H < R < L$, where R is the stabiliser in L of a hyperbolic line. Moreover, the action of R on the set of cosets of N_2 is equivalent to the primitive action of $O_6^+(q)$ on decompositions of a 6-dimensional vector space into complementary totally singular 3-spaces. We have already seen in Section 6.2 that this action of R has a subdegree divisible by p and so by Lemma 2.1(ii), L , and hence G , has a subdegree divisible by p . Since p divides $|\Omega|$ it follows that G is not $\frac{3}{2}$ -transitive.

For H in case (5), by [35, Proposition 3.3.1], $H \cap L = N_3$. If p divides $|H|$, either $p = 2$, or p is odd and H contains an outer automorphism of order p . Suppose first that $p = 2$ and let $x \in H \cap L$ be an involution. Then by Proposition 5.7 and the fact that x is not of b -type, $|x^G| > \frac{1}{4}q^{10}$. Since $H \cap L$ contains at most $(q^2 + 1)^2 \cdot 4$ involutions it follows from Lemma 2.6(ii) that for $q \geq 8$, G has a subdegree divisible by $p = 2$. For $q = 4$, a MAGMA [7] calculation shows that $H \cap L$ contains only 391 involutions while there is an involution $x \in H \cap L$ such that $|x^L| = 16707600$. Lemma 2.6(ii) then yields a subdegree divisible by p . For $q = 2$, a similar calculation reveals that $H \cap L$ contains 55 involutions and contains an involution x such that $|x^L| = 3780$. Again, Lemma 2.6(ii) yields a subdegree divisible by p . Next suppose that p is odd and H contains an outer automorphism x of order p . Then by Lemma 5.8, $|x^G| > \frac{1}{8}q^{14}$, while $|x^G \cap H| < |L \cap H|(p - 1) \leq 4q(q^2 + 1)^2$. Thus Lemma 2.6(ii) implies that G has a subdegree divisible by p . Since $|\Omega|$ is divisible by p this implies that if $|H|$ is divisible by p then G is not $\frac{3}{2}$ -transitive.

If $|H|$ is coprime to p (still with H as in (5)) then p is odd, and $p \neq 3$ as H contains a triality. Let $x \in H \cap L$ be an involution. We see in the proof of [35, Proposition 3.3.1] that N_3 is a subgroup of $(\Omega_4^-(q) \times \Omega_4^-(q)) : 2^2$, the stabiliser in L of a pair $\{W, W^\perp\}$ for some elliptic 4-space W . Thus we can choose x such that x acts nontrivially on both W and W^\perp and so $\nu(x) \geq 2$. Then by [9, Proposition 3.37], $|x^G| > \frac{1}{4(q+1)}q^{13}$. Now $|H \cap L| = (q^2 + 1)^2 \cdot 4$ and so as $p > 3$, Lemma 2.6(ii) implies that G has an even subdegree. Since $|\Omega|$ is even, it follows that G is not $\frac{3}{2}$ -transitive.

Finally suppose that H is as in case (6). Here $L = \text{P}\Omega_8^+(p)$ with $p \geq 3$ a prime and $H \cap L = [2^9] \rtimes \text{PSL}_3(2)$. It follows from [34, Proposition 3.4.2] that $H \cap L < R < L$ where $R = \text{P}\Omega_8^+(2)$. A MAGMA [7] calculation shows that the subdegrees of R acting on the cosets of $H \cap L$ in R are

$$\{1, 14^2, 28, 168^2, 224, 448^2, 512\}$$

and by Lemma 2.1(ii) these are also subdegrees of L acting on Ω . Thus G is not $\frac{3}{2}$ -transitive. Moreover, if p divides $|H|$ then $p = 3$ or 7 and G has a subdegree divisible by p .

7. EXCEPTIONAL GROUPS OF LIE TYPE

In this section we prove Theorems 1.2 and 1.3 for almost simple groups of exceptional Lie type.

Theorem 7.1. *Let G be an almost simple group with socle an exceptional group of Lie type in characteristic p . Suppose that G acts primitively on a set Ω with point stabiliser H , and assume p divides $|H|$. Then one of the following holds:*

- (1) G has a subdegree which is divisible by p .
- (2) $L = G_2(2)'$, $|\Omega| = 28$ and $H = N_G(3^{1+2})$. (Here the subdegrees are 1, 27.)
- (3) $L = G_2(2)'$, $|\Omega| = 36$, $H = \text{PSL}_3(2)$ and $G = L$. (Here the subdegrees are 1, 7, 7, 21.)
- (4) $L = {}^2G_2(3)'$, $|\Omega| = 9$, $H = N_G(2^3)$ and $G = {}^2G_2(3)$. (Here the subdegrees are 1, 8.)

Theorem 7.2. *If G is an almost simple group of exceptional Lie type which is $\frac{3}{2}$ -transitive but not 2-transitive on a set Ω , then $G = {}^2G_2(3)'$ of degree 28.*

Note that ${}^2G_2(3)' = \text{PSL}_2(8)$ so this case is recorded in part (iii) of Theorem 1.2.

In this section we will use Lie notation for classical groups instead of the notation used previously, for example $L_n(q)$ denotes $\text{PSL}_n(q)$ and $U_n(q)$ denotes $\text{PSU}_n(q)$. We also use Alt_n and Sym_n to denote

TABLE 10. Bounds for unipotent class sizes in exceptional groups

G	bounds for $ u_\alpha^G $	bounds for $ u^G $
$E_8(q)$	$q^{58} < u_\alpha^G < 2q^{58}$	$ u^G > q^{92}$
$E_7(q)$	$q^{34} < u_\alpha^G < 2q^{34}$	$ u^G > q^{52}$
$E_6^\epsilon(q)$	$q^{19}(q^3 - 1) < u_\alpha^G < 2q^{22}$	$ u^G > q^{31}$
$F_4(q)$	$q^{16} < u_\alpha^G < 2q^{16}$	$ u^G > q^{21}$
$G_2(q)$	$ u_\alpha^G = q^6 - 1$	$ u^G \geq (q^6 - 1)(q^2 - 1)$
${}^2F_4(q)$	$ u_\alpha^G = (q^6 + 1)(q^3 + 1)(q^2 - 1)$	$ u^G > q^{13}(q - 1)$
${}^3D_4(q)$	$ u_\alpha^G > q^8(q^2 - 1)$	$ u^G > q^{16}$
${}^2G_2(q)$	$ u_\alpha^G = (q^3 + 1)(q - 1)$	—
${}^2B_2(q)$	$ u_\alpha^G = (q^2 + 1)(q - 1)$	—

TABLE 11. Bounds for class sizes of long root elements in classical groups

G	$ u_\alpha^G \leq$
$A_n^\epsilon(q)$	$2q^{2n}$
$B_n(q)$	$2q^{4n-4}$
$C_n(q)$	q^{2n}
$D_n^\epsilon(q)$	$2q^{4n-6}$

the alternating and symmetric group of degree n to avoid confusion with the group $A_n(q)$ of Lie type. Moreover, for a group G of Lie type we use $W(G)$ to denote the Weyl group of G and G_0 to denote the socle of G . For a finite group X , $\text{soc}(X)$ denotes the socle of X .

7.1. Preliminary lemmas. As the simple groups $G_2(2)'$ and ${}^2G_2(3)'$ are the classical groups $\text{PSU}_3(3)$ and $\text{PSL}_2(8)$ respectively, we have already considered them in the analysis for $\frac{3}{2}$ -transitive groups. The assertions about their subdegrees divisible by p can be easily verified. From now on we will exclude the simple groups $G_2(2)'$ and ${}^2G_2(3)'$. In the next few results we also exclude ${}^2F_4(2)'$ – this is dealt with separately in Lemma 7.7.

The next two lemmas give bounds for the sizes of certain unipotent classes in groups of Lie type. The first follows from the determination of unipotent classes in exceptional groups of Lie type (see [54, 55], and [47] for complete information), and the second from elementary calculations in classical groups.

Lemma 7.3. *Let $G = G(q)$, $q = p^a$ be a simple exceptional group of Lie type. Let u_α be a long root element of G , and let u be a non-identity unipotent element of G which is not a long root element (or a short root element when $(G, p) = (F_4(q), 2)$ or $(G_2(q), 3)$). Then bounds for the sizes of the classes u_α^G and u^G are given in Table 10.*

Lemma 7.4. *Let $G = G(q)$, $q = p^a$ be a simple classical group, and let u_α be a long root element of G . Upper bounds for the size of the class u_α^G are given in Table 11. Moreover, if $p = 2$ and $G = \text{SO}_{2n}^\epsilon(q)$ ($n \geq 2$, $\epsilon = \pm$), then the number of reflections in G is $q^{n-1}(q^n - \epsilon)$.*

Lemma 7.5. *Let $G = G(q)$, $q = p^a$ be a simple exceptional group of Lie type. Suppose $p > 2$ and there is an outer automorphism ϕ of G of order p . Then one of the following holds:*

- (i) ϕ is a field automorphism with centralizer $G(q^{1/p})$;
- (ii) $G = {}^3D_4(q)$, $p = 3$ and ϕ is a graph automorphism with centralizer $G_2(q)$ or $q^5.A_1(q)$.

Proof. This follows from [27, Section 4.9]. □

We shall need the following result concerning the maximal subgroups of exceptional groups of Lie type. This is an amalgamation of results from several papers, taken from [50, Theorem 8], where references can be found. In part (vii), \bar{G} denotes a simple algebraic group over $\overline{\text{GF}(q)}$ of the same type as G , and σ a Frobenius morphism of \bar{G} such that $G_0 = \bar{G}'_\sigma$.

Lemma 7.6. *Let G be an almost simple group with socle $G_0 = G(q)$ ($q = p^a$) an exceptional group of Lie type over $\text{GF}(q)$, and let H be a maximal subgroup of G . Then one of the following holds.*

TABLE 12. Socles in Lemma 7.6(iii)

G_0	possibilities for $\text{soc}(H)$
$G_2(q)$	$A_1(q) (p \geq 7)$
${}^3D_4(q)$	$G_2(q), A_2^\epsilon(q)$
$F_4(q)$	$A_1(q) (p \geq 13), G_2(q) (p = 7), A_1(q)G_2(q) (p \geq 3, q \geq 5)$
$E_6^\epsilon(q)$	$A_2^\epsilon(q) (p \geq 5), G_2(q) (p \neq 7), C_4(q) (p \geq 3), F_4(q),$ $A_2^\epsilon(q)G_2(q) ((q, \epsilon) \neq (2, -))$
$E_7(q)$	$A_1(q) (2 \text{ classes}, p \geq 17, 19), A_2^\epsilon(q) (p \geq 5), A_1(q)A_1(q) (p \geq 5),$ $A_1(q)G_2(q) (p \geq 3, q \geq 5), A_1(q)F_4(q) (q \geq 4), G_2(q)C_3(q)$
$E_8(q)$	$A_1(q) (3 \text{ classes}, p \geq 23, 29, 31), B_2(q) (p \geq 5), A_1(q)A_2^\epsilon(q) (p \geq 5),$ $G_2(q)F_4(q), A_1(q)G_2(q)G_2(q) (p \geq 3, q \geq 5),$ $A_1(q)G_2(q^2) (p \geq 3, q \geq 5)$

TABLE 13. Subdegrees of primitive actions of ${}^2F_4(2)'$

Degree	Subdegrees
351	126, 224
351	126, 224
364	12, 108, 243
364	12, 108, 243
378	52, 117, 208
378	52, 117, 208
2808	56, 63, 84^2 , 252^2 , 504, 1512
3159	14, 64, 168, 224^2 , 448^4 , 672
3888	78^2 , 91^3 , 182^3 , 364^2 , 546^2 , 1092
7371	18, 32^2 , 64, 72^3 , 96^4 , 144^2 , 288^{14} , 576^4

- (i) H is parabolic.
- (ii) H is a subgroup of maximal rank, given by [46].
- (iii) $\text{soc}(H)$ is as in Table 12.
- (iv) $H = G(q_0)$, a subgroup of the same type as G (possibly twisted), over a subfield $\text{GF}(q_0)$ of $\text{GF}(q)$.
- (v) H is a local subgroup, given by [21, Theorem 1].
- (vi) $G_0 = E_8(q)$, $p > 5$, and $H \cap G_0 = \text{PGL}_2(q) \times \text{Sym}_5$ or $(\text{Alt}_5 \times \text{Alt}_6).2^2$.
- (vii) $G_0 = E_8(q)$, $E_7(q)$, $E_6^\epsilon(q)$ or $F_4(q)$, and $\text{soc}(H) = H(r)$, a group of Lie type over $\text{GF}(r)$, where $r = p^b$. Moreover $\text{rank}(H(r)) \leq \frac{1}{2}\text{rank}(G)$; and either $r \leq 9$, or $H(r) = A_2^\pm(16)$, or $H(r) \in \{A_1(r), {}^2G_2(r), {}^2B_2(r)\}$. Finally, $H(r)$ is not of the form M'_σ , where M is a σ -stable subgroup of positive dimension in \bar{G} .
- (viii) $\text{soc}(H)$ is a simple group that is not a group of Lie type of characteristic p , and the possibilities for $\text{soc}(H)$ are given by [49].

7.2. Classifying p -subdegrees. In this section we prove Theorem 7.1. Let G be an almost simple group with socle $G_0 = G(q)$ ($q = p^a$) an exceptional group of Lie type in characteristic p . Let G act primitively on a set Ω , let $H = G_\alpha$ where $\alpha \in \Omega$, and suppose that p divides $|\Omega|$. Now H is a maximal subgroup of G . We treat the various possibilities for H given by Lemma 7.6.

We first deal with ${}^2F_4(2)'$.

Lemma 7.7. *Theorem 7.1 holds when $G_0 = {}^2F_4(2)'$.*

Proof. The subdegrees for all the primitive actions of ${}^2F_4(2)'$ were determined by GAP [25] calculations and are given in Table 13. In each case there is an even subdegree. By [22, p74] the only maximal subgroup of ${}^2F_4(2)$, not arising from a maximal subgroup of ${}^2F_4(2)'$ is $C_{13} \rtimes C_{12}$. The subdegrees of this action are 1, 13, 26, 39^{10} , 78^{37} , 52^8 and 156^{1453} . \square

From now on assume that $G_0 \neq {}^2F_4(2)'$. In view of Lemma 2.6(ii), in proving Theorem 7.1 we may assume that for any non-identity p -element $u \in H$, we have

$$(2) \quad |u^G| < |u^G \cap H|^2.$$

TABLE 14. Subgroups of maximal rank

G_0	possibilities for H^0
$E_8(q)$	$A_1(q)E_7(q), D_8(q), A_8^\epsilon(q), A_2^\epsilon(q)E_6^\epsilon(q),$ $D_4(q)^2, D_4(q^2), {}^3D_4(q)^2, {}^3D_4(q^2),$ $A_4^\epsilon(q)^2, {}^2A_4(q^2), A_2^\epsilon(q)^4, {}^2A_2(q^2)^2,$ ${}^2A_2(q^4), A_1(q)^8$
$E_7(q)$	$E_6^\epsilon(q) \cdot (q - \epsilon), A_1(q)D_6(q), A_7^\epsilon(q), A_2^\epsilon(q)A_5^\epsilon(q),$ $A_1(q)^3 D_4(q), A_1(q^3) {}^3D_4(q), A_1(q)^7, A_1(q^7)$
$E_6^\epsilon(q)$	$A_1(q)A_5^\epsilon(q), A_2^\epsilon(q)^3, A_2(q^2)A_2^{-\epsilon}(q), A_2^\epsilon(q^3),$ $D_4(q) \cdot (q - \epsilon)^2, {}^3D_4(q) \cdot (q^2 + \epsilon q + 1), D_5^\epsilon(q) \cdot (q - \epsilon)$
$F_4(q)$	$A_1(q)C_3(q), B_4(q), D_4(q), {}^3D_4(q), A_2^\epsilon(q)^2$ $B_2(q)^2 (q \text{ even}), B_2(q^2) (q \text{ even})$
$G_2(q)$	$A_1(q)^2, A_2^\epsilon(q)$
${}^2F_4(q) (q > 2)$	${}^2A_2(q), {}^2B_2(q)^2, B_2(q)$
${}^3D_4(q)$	$A_1(q)A_1(q^3), A_2^\epsilon(q) \cdot (q^2 + \epsilon q + 1)$
${}^2G_2(q) (q > 3)$	$A_1(q) \times 2$

Lemma 7.8. *Theorem 7.1 holds if H is a parabolic subgroup of G .*

Proof. This holds by Lemma 2.1(iv). □

The next two lemmas deal with the proof of Theorem 7.1 in the case where H is a subgroup of maximal rank, as in (ii) of Lemma 7.6. The lists of maximal subgroups of maximal rank can be found in Tables 5.1 and 5.2 of [46]: the subgroups in Table 5.2 are normalizers of maximal tori in G , and those in Table 5.1 are not. It is convenient to handle these cases separately.

Lemma 7.9. *Theorem 7.1 holds if H is a subgroup of maximal rank which is not the normalizer of a maximal torus.*

Proof. Here H is as in [46, Table 5.1]. We list the possibilities for H in Table 14; for notational convenience each subgroup is recorded via a subgroup H^0 of small index in H .

Suppose first that $G_0 \neq {}^2F_4(q)$ or ${}^2G_2(q)$ and H^0 does not contain a long root element of G . The only such cases are:

$$\begin{aligned}
 (3) \quad G_0 = E_8(q) : & \quad {}^2A_4(q^2), D_4(q^2), {}^3D_4(q^2), {}^2A_2(q^2)^2, {}^2A_2(q^4) \\
 & E_7(q) : \quad A_1(q^7) \\
 & E_6^\epsilon(q) : \quad A_2^\epsilon(q^3) \\
 & F_4(q) : \quad B_2(q^2) (p = 2)
 \end{aligned}$$

All of these cases are easily dealt with as follows. For the $E_8(q)$ or $E_7(q)$ cases observe that for a non-root unipotent element $u \in H^0$ we have $|u^G| > q^{92}$ or q^{52} respectively by Lemma 7.3, and so we have $|u^G| > |u^G \cap H|^2$ even if we use the total number of unipotent elements of H as an upper bound for $|u^G \cap H|$; the conclusion follows by Lemma 2.6(ii). For the $E_6^\epsilon(q)$ case, observe that $H^0 = A_2^\epsilon(q^3)$ arises from a subsystem subgroup A_2^3 of the algebraic group E_6 , and hence a transvection u in H^0 lies in the unipotent class labelled by the Levi $3A_1$ in E_6 , and we have $|u^G| > q^{40}$ (see [54]), hence again $|u^G| > |u^G \cap H|^2$. And for the $F_4(q)$ case the same argument applies, using an element $u \in H^0$ in the class $A_1\tilde{A}_1$ of F_4 (for which $|u^G| > q^{28}$ by [65]).

Suppose now that none of the cases in (3) holds (still excluding $G_0 = {}^2F_4(q)$ or ${}^2G_2(q)$). Then H^0 contains a long root element u_α of G . Using [39, 1.13], we see that if u is any element of $u_\alpha^G \cap H$ then one of the following holds:

- (i) u is a long root element of one of the quasisimple factors of H^0
- (ii) $p = 2$, H^0 has a factor $D_n^\epsilon(q)$, and u is a reflection in a subgroup $D_n^\epsilon(q).2$ of H .

Lower bounds for $|u_\alpha^G|$ are given by Lemma 7.3, and upper bounds for $|u_\alpha^G \cap H|$ follow from Lemmas 7.3 and 7.4. We find from these that

$$|u_\alpha^G| > |u_\alpha^G \cap H|^2,$$

except in the following cases:

$$\begin{aligned} G_0 &= E_8(q), H^0 = A_1(q)E_7(q) \\ G_0 &= E_7(q), H^0 = E_6^\epsilon(q) \cdot (q - \epsilon) \text{ or } A_1(q)D_6(q) \\ G_0 &= E_6^\epsilon(q), H^0 = D_5^\epsilon(q) \cdot (q - \epsilon) \\ G_0 &= F_4(q), H^0 = B_4(q), D_4(q) \text{ or } {}^3D_4(q) \\ G_0 &= G_2(q), H^0 = A_2^\epsilon(q) \end{aligned}$$

Hence by Lemma 2.6(ii) we may assume that one of these cases holds.

Consider the case where $G_0 = E_8(q)$, $H^0 = A_1(q)E_7(q)$. If we let T be a subsystem subgroup ${}^2E_6(q)$ of H , then $C_G(T)$ contains a subgroup ${}^2A_2(q)$ not lying in H , so $N_G(T) \not\leq H$. Also for any Sylow p -subgroup S of H , $\langle T, S \rangle$ contains the factor $E_7(q)$ of H^0 by Lemma 2.3. Hence G has a subdegree divisible by p , by Lemma 2.2.

Next let $G_0 = E_7(q)$, $H^0 = E_6^\epsilon(q) \cdot (q - \epsilon)$ or $A_1(q)D_6(q)$. In the latter case we take T to be a subgroup ${}^2A_5(q)$ of the D_6 factor; then $C_G(T)$ contains a subgroup ${}^2A_2(q)$ not lying in H , and the argument of the previous paragraph applies. And when $H^0 = E_6^\epsilon(q) \cdot (q - \epsilon)$, define T to be a subgroup $F_4(q)$ of H^0 , and note that $C_G(T)$ contains a subgroup $A_1(q)$ (see [48, 4.6]), whereas $C_H(T)$ does not, provided $(q, \epsilon) \neq (2, -)$. Thus with this exception, Lemma 2.2 again gives the conclusion. If $(q, \epsilon) = (2, -)$, then $G = E_7(2)$, $H = 3 \cdot {}^2E_6(2).S_3$. Choose a subsystem subgroup $T = D_4(2)$ of H . From [46, Table 5.1] we see that $|N_H(T)/T| = 2^23^4$, while $|N_G(T)/T| = 2^43^4$. Hence $N_G(T) \not\leq H$. Moreover T lies in no parabolic subgroup of H , and hence $\langle S, T \rangle$ contains $3 \cdot {}^2E_6(2)$ for any Sylow 2-subgroup S of H , by Lemma 2.3. Now Lemma 2.2 gives the conclusion.

Now let $G_0 = E_6^\epsilon(q)$, $H^0 = D_5^\epsilon(q) \cdot (q - \epsilon)$. For $\epsilon = -$ take T to be a maximal torus of order $(q + 1)^6$ in H (or of index $(3, q + 1)$ in this), and apply Lemma 2.2 with the characteristic p . Now suppose $\epsilon = +$, $H^0 = D_5(q) \cdot (q - 1)$. If q is odd then $H = C_G(t)$ for some involution $t \in G$. There is a subgroup $D = D_4(q)$ of H such that $Z(D) = \langle t, u \rangle$, where u is a conjugate of t . Then $C_H(t, u)$ is a 2-point stabiliser and the subdegree $|H : C_H(t, u)|$ is divisible by p , as required. If q is even, note that G contains a graph automorphism, since otherwise H lies in a parabolic subgroup. Let T be a maximal torus in H of order $(q^5 - 1)(q - 1)$ (or of index $(3, q - 1)$ in this), lying in an A_5 Levi subgroup. This torus is not normalized by a graph involution of $D_5(q)$, whereas $N_G(T)/T$ does contain an involution (see [19]). Hence $N_G(T) \not\leq H$, and so T lies in a 2 point stabiliser $H \cap H^g$ for some $g \in G$. If the subdegree $|H : H \cap H^g|$ is odd, then $H \cap H^g$ must contain the derived subgroup of an $A_4(q)$ -parabolic of H . However such a parabolic is not normalized by a graph involution of $D_5(q)$, so since H contains such a graph involution, it follows that $|H : H \cap H^g|$ is even anyway.

When $G_0 = F_4(q)$, $H^0 = B_4(q)$, $D_4(q)$ or ${}^3D_4(q)$, we take $T = (q + 1)^4$, ${}^2A_2(q)$ or $A_2(q)$ respectively. In the first case T is a maximal torus and $N_G(T)$ induces $W(F_4)$ on T , so $N_G(T) \not\leq H$; and in the second and third cases $N_G(T)$ contains ${}^2A_2(q)^2$ or $A_2(q)^2$, so again $N_G(T) \not\leq H$. Now the conclusion follows from Lemma 2.2.

When $G_0 = G_2(q)$, $H^0 = A_2^\epsilon(q)$, the subdegrees are given by [41, Proposition 1] for $\epsilon = -$, and by [42, 6.8] for $\epsilon = +$, and there is a subdegree divisible by p .

Finally, we need to handle the cases where $G_0 = {}^2F_4(q)$ or ${}^2G_2(q)$. Consider the first case ${}^2F_4(q)$. Here $q \geq 8$ (as we have already dealt with the $q = 2$ case), and $H^0 = {}^2A_2(q)$, ${}^2B_2(q)^2$ or $B_2(q)$. Let u be an involution in H^0 , and when $H^0 \neq {}^2A_2(q)$ take u to be a non-root involution. Then by Lemma 7.4 we have $|u^G| > q^9(q^2 - 1)$, and also $|u^G| > q^{13}(q - 1)$ when $H^0 \neq {}^2A_2(q)$. If $i_2(H)$ denotes the number of involutions in H , then by [38, 1.3] we have $i_2(H) < 2(q^5 + q^4)$, $4(q^3 + q^2)^2$ or $2(q^6 + q^5)$ in the respective cases for H^0 . Hence we see that $|u^G| > i_2(H)^2 \geq |u^G \cap H|^2$, giving the conclusion by Lemma 2.6(ii). Finally, if $G_0 = {}^2G_2(q)$, $H^0 = 2 \times L_2(q)$, then for an element $u \in H^0$ of order 3 we have $|u^G| = \frac{1}{2}q(q^3 + 1)(q - 1)$ (see [67]), while $|u^G \cap H| = q^2 - 1$, so again Lemma 2.6(ii) gives the result. \square

Lemma 7.10. *Theorem 7.1 holds if H is the normalizer of a maximal torus.*

Proof. Here H is as in [46, Table 5.2]. If p does not divide $|H \cap G_0|$, then H contains an outer automorphism u of order p , and bounds for $|u^G|$ are given by Lemma 7.5. Otherwise, p divides $|H \cap G_0|$ (and in particular p divides $|W(G)|$), and it is clear from the action on the adjoint module $L(G)$ that H contains a non-identity p -element u which is not a long root element of G ; bounds for $|u^G|$ are given in Lemma 7.3. We may assume by Lemma 2.6(ii) that $|H| > |u^G|^{1/2}$, and so from the above bounds, we see that H

is as in the following table (recall that we are assuming that $G_0 \neq {}^2F_4(2)'$ in view of Lemma 7.7):

G_0	$H \cap G_0$	q
$E_7(q)$	$3^7.W(E_7)$	$q = 2$
$E_6^\epsilon(q)$	$((q+1)^6/(3, q+1)).W(E_6)$	$q = 2$ or $3, \epsilon = -$
	$7^3.(3^{1+2}.\text{SL}_2(3))$	$q = 2, \epsilon = +$
$F_4(q)$	$7^2.(3 \times \text{SL}_2(3))$	$q = 2$
${}^2B_2(q)$	$(q-1).2, (q \pm \sqrt{2q} + 1).4$	

Consider the case where $G_0 = E_7(q), q = 2$ and $H = 3^7.W(E_7)$. Here H has an element x of order 8. Inspection of [55] shows that the smallest class of elements of order 8 in G is the class labelled $D_4(a_1)$, which has size greater than $q^{94}/6$. However $|H| = 3^7|W(E_7)| < 2^{34} < 2^{47}/\sqrt{6}$, a contradiction.

The cases in the table with $G_0 = E_6^\epsilon(q)$ are handled similarly, using an element of order 8 in H (if $q = 2, \epsilon = -$), an element of order 4 (if $q = 2, \epsilon = +$), an element of order 9 (if $q = 3$), and [54] for the classes of G . Likewise, for $G = F_4(q), q = 2$, we use an element of order 4 in H , together with [22] for the classes.

Finally, in the case where $G_0 = {}^2B_2(q)$, pick an element u of order 2 or 4 in H and observe using 7.3 that $|u^G \cap H| < |u^G|^{1/2}$. This completes the proof. \square

Lemma 7.11. *Theorem 7.1 holds if H is as in Lemma 7.6(iii).*

Proof. Here $H_0 = \text{soc}(H)$ is as in Table 12. In all cases H contains a unipotent element u which is not a long root element. By Lemma 7.3 and our assumption that $|u^G| < |u^G \cap H|^2 < |H|^2$, we see that H is as in the following table:

G_0	possibilities for H_0
$E_8(q)$	$G_2(q)F_4(q)$
$E_7(q)$	$G_2(q)C_3(q), A_1(q)F_4(q)$
$E_6^\epsilon(q)$	$F_4(q), C_4(q) (q \text{ odd}), A_2^\epsilon(q)G_2(q)$
$F_4(q)$	$A_1(q)G_2(q) (q \text{ odd}), G_2(q) (p = 7)$
${}^3D_4(q)$	$G_2(q), A_2^\epsilon(q)$

Suppose first that $G_0 \neq {}^3D_4(q)$. The cases $(G_0, H_0) = (E_6^\epsilon(q), C_4(q)) (p > 2)$ and $(F_4(q), G_2(q)) (p = 7)$ are covered by Lemma 2.4. In all other cases H contains a long root element u_α of G , and using [39, 1.13], we see that $u_\alpha^G \cap H$ consists of long root elements of H . Now a check using Lemma 7.3 shows that, excluding the case $(E_6^\epsilon(q), F_4(q))$, we have $|u_\alpha^G| > |u_\alpha^G \cap H|^2$, a contradiction. In the exceptional case $G_0 = E_6^\epsilon(q), H_0 = F_4(q)$, and we take a subsystem subgroup $T = {}^3D_4(q)$ of H . Then T is centralized by an element of order $q^2 + \epsilon q + 1$ in G not in H , so $N_G(T) \not\leq H$. Now the conclusion follows from Lemma 2.2.

Suppose finally that $G_0 = {}^3D_4(q)$. If $H_0 = G_2(q)$, let T be a subsystem subgroup $\text{SL}_3(q)$ of H_0 . Then $N_G(T) \not\leq H$ as G has an element of order $q^2 + q + 1$ centralizing T , so the result follows from Lemma 2.2. Now consider $H_0 = A_2^\epsilon(q)$. We may assume that $p = 2$ by Lemma 2.4. Let $u \in H_0$ be an involution which is not a long root element of G , so that $|u^G| > q^{16}$ by Lemma 7.3. Then $|u^G \cap H|$ is certainly no more than the total number of involutions in H_0 , which is at most $2(q^5 + q^4)$ by [38, 1.3]. Hence again $|u^G| > |u^G \cap H|^2$. \square

Lemma 7.12. *Theorem 7.1 holds if H is as in Lemma 7.6(iv).*

Proof. Here H has socle $H_0 = G(q_0)$, a group of the same type as G (possibly twisted) over a subfield $\text{GF}(q_0)$ of $\text{GF}(q)$. We take T to be a maximal torus of H as in the following table (for the existence of T , see [18]). The table also gives a primitive prime divisor $(q_0)_r$ of $q_0^r - 1$ that divides $|T|$:

H_0	$ T $	$(q_0)_r$ dividing $ T $
$E_8(q_0)$	$q_0^8 - q_0^7 + q_0^5 - q_0^4 + q_0^3 - q_0 + 1$	$(q_0)_{30}$
$E_7(q_0)$	$(q_0^6 - q_0^3 + 1)(q_0 + 1)$	$(q_0)_{18}$
$E_6^\epsilon(q_0)$	$q_0^6 + \epsilon q_0^3 + 1$	$(q_0)_9 (\epsilon = +), (q_0)_{18} (\epsilon = -)$
$F_4(q_0)$	$q_0^4 - q_0^2 + 1$	$(q_0)_{12}$
$G_2(q_0)$	$q_0^2 + q_0 + 1$	$(q_0)_3$
${}^2F_4(q_0)'$	$q_0^2 + \sqrt{2q_0^3} + q_0 + \sqrt{2q_0} + 1$	$(q_0)_{12}$
${}^3D_4(q_0)$	$q_0^4 - q_0^2 + 1$	$(q_0)_{12}$
${}^2G_2(q_0)$	$q_0 + \sqrt{3q_0} + 1$	$(q_0)_6$
${}^2B_2(q_0)$	$q_0 + \sqrt{2q_0} + 1$	$(q_0)_4$

In all cases T lies in a maximal torus of G which centralizes it, the order of which is given by [18], and hence we see that $C_G(T) \not\leq H$ with the following exceptions: $H_0 = {}^2G_2(3) < G_2(3)$ and $H_0 = {}^2F_4(2) < F_4(2)$. Moreover the divisor $(q_0)_r$ of $|T|$ shows that T is not contained in any parabolic subgroup of H_0 . Hence, apart from in the above exceptional cases, the conclusion follows from Lemma 2.2. As for the exceptional cases: when $H_0 = {}^2G_2(3) < G_2(3)$, take $T = 2^3$, a Sylow 2-subgroup of H_0 , and apply Lemma 2.2; and when $H_0 = {}^2F_4(2) < F_4(2)$, take $T = 3^{1+2}$, a Sylow 3-subgroup of H_0 . \square

Lemma 7.13. *Theorem 7.1 holds if H is as in Lemma 7.6(v) or (vi).*

Proof. Here H is either a local subgroup given by [21, Theorem 1], or one of the subgroups in 7.6(vi). If u is an element of order p in H , it is easy using Lemma 7.3 to check that $|H|^2 < |u^G|$ except in the following cases:

G_0	$H \cap G_0$
$G_2(3)$	$2^3.\text{SL}_3(2)$
${}^2E_6(2)$	$U_3(2) \times G_2(2)$
$E_7(3)$	$L_2(3) \times F_4(3)$
$E_7(q)$ (q odd)	$(2^2 \times D_4(q).2^2).S_3$

The last case is dealt with using Lemma 2.4, and the second and third cases are handled as in the proof of Lemma 7.11.

When $G_0 = G_2(3)$ and $H \cap G_0 = 2^3.\text{SL}_3(2)$, a GAP [25] calculation shows that the subdegrees for G_0 are 14, 64, 168, 224^2 , 448^4 and 672 . \square

Lemma 7.14. *Theorem 7.1 holds if H is as in Lemma 7.6(vii).*

Proof. Here G_0 is $E_8(q)$, $E_7(q)$, $E_6^\epsilon(q)$ or $F_4(q)$, and $\text{soc}(H) = H(r)$, a group of Lie type over $\text{GF}(r)$, where $r = p^b$. Moreover $\text{rank}(H(r)) \leq \frac{1}{2}\text{rank}(G)$; and either $r \leq 9$, or $H(r) = A_2^\pm(16)$, or $H(r) \in \{A_1(r), {}^2G_2(r), {}^2B_2(r)\}$. Regard G_0 as \bar{G}_σ' , where σ is a Frobenius morphism of the simple algebraic group \bar{G} over $\text{GF}(q)$ of the same type as G .

Assume that p is odd. Now $|G : H|$ is even by [43], so by Lemma 2.4 we may assume that $H(r)$ is $A_2(r)$ or $A_4(r)$ (the latter only if $G_0 = E_8(q)$). Moreover $H(r)$ contains no long root element of G by [48], so if $1 \neq u \in H(r)$ is a unipotent element, then our assumption $|u^G \cap H|^2 > |u^G|$, together with Lemma 7.3, leaves only the possibility

$$H(r) = A_2(9) < F_4(3) = G_0.$$

However $A_2(9)$ has an element of order $3^4 + 3^2 + 1$, whereas $F_4(3)$ has no torus divisible by this number (see [18]), so $A_2(9) \not\leq F_4(3)$.

Now consider $p = 2$. Suppose first that $H(r)$ contains a long root element u_α of G . If $r > 2$ we argue as in [38, p.437] (fourth paragraph) that there is a subgroup M of positive dimension in \bar{G} such that $H(r) = M'_\sigma$, contrary to Lemma 7.6(vii). Hence $r = 2$ and $\text{soc}(H) = H(2)$. Moreover, elements of $u_\alpha^G \cap H(2)$ are root elements of $H(2)$ by [66, Theorem 1], so using Lemma 7.3 we see that $|u_\alpha^G \cap H(2)|^2 < |u_\alpha^G|$, contrary to assumption.

Finally, assume that $p = 2$ and $H(r)$ contains no long root element of G . We may assume that $G_0 \neq F_4(2)$ by [59]. Let $u \in H(r)$ be an involution. Then $|u^G \cap H(r)| \leq i_2(H(r))$, the number of involutions in $H(r)$, so by assumption we have $i_2(H(r)) > |u^G|^{1/2}$. By [38, 1.3] we have $i_2(H) < 2(r^M + r^{M-1})$, where $M = \dim \bar{G} - N$, N being the number of positive roots in the root system of \bar{G} (and M is half this number when $H(r)$ is of type 2F_4 , 2G_2 or 2B_2). Also lower bounds for $|u^G|$ by Lemma 7.3. One now checks that the only possibilities for $H(r)$ satisfying the inequality $i_2(H) > |u^G|^{1/2}$ and also having order dividing $|G|$ are as follows:

G_0	$H(r)$
$E_8(2)$	$C_4(8), D_4^\epsilon(8)$
$E_7(2)$	$C_3(8), A_3^\epsilon(8)$
$E_6^\epsilon(2)$	$A_3^\epsilon(8), A_3(4), C_2(8), A_2(16), A_2^\epsilon(8), G_2(4)$
$E_6^\epsilon(4)$	$C_3(8)$
$F_4(4)$	$A_2^\epsilon(16)$
$F_4(q)$	$A_1(q^6)$

In most of these cases it is easy to use [18] to produce an element of large order in $H(r)$ which does not divide the order of a maximal torus of G_0 : for example, $D_4^\epsilon(8)$ has an element of order $2^9 + 1$, so cannot

lie in $E_8(2)$, and so on. The possibilities which do not succumb to this argument are:

$$\begin{aligned} G_0 &= E_6^\epsilon(2) : H(r) = A_2^\epsilon(8), G_2(4) \\ G_0 &= F_4(4) : H(r) = A_2^-(16) \end{aligned}$$

For $G_0 = E_6^\epsilon(2)$ and $H(r) = A_2^\epsilon(8)$ we calculate $i_2(\text{Aut}(H(r)))$ precisely and check that it is less than $2^{31/2}$, hence less than $|u^G|^{1/2}$. And for $H(r) = G_2(4)$, observe that $H(r)$ has a subgroup $\text{SL}_3(4)$; this centralizes a 3-element of G_0 , from which we see that it is a subsystem group, hence contains long root elements of G_0 , contrary to assumption. The case with $G_0 = F_4(4)$ does not arise as $U_3(q^2) \not\leq F_4(q)$ by [46, 4.5]. This completes the proof. \square

Lemma 7.15. *Theorem 7.1 holds if H is as in Lemma 7.6(viii).*

Proof. Here $H_0 = \text{soc}(H)$ is a simple group not in $\text{Lie}(p)$. The possibilities for H_0 are given by [49, Theorem 2]. In every case one checks that all prime divisors of $|\text{Out}(H_0)|$ also divide $|H_0|$; hence the fact that p divides $|H|$ implies that p divides $|H_0|$.

Suppose first that H_0 contains a long root element u_α of G . Then $p = 2$ by [48, 6.1]. Theorem 1 of [66] gives a list of possible isomorphism types for H_0 , and identifies u_α as a root involution for each type. Combining this with [49], we see that $H_0 = A_6, J_2$ or Fi_{22} and $|u_\alpha^G \cap H| = 45, 315$ or 3510 respectively. Now the bound $|u_\alpha^G \cap H|^2 > |u^G|$, together with Lemma 7.3, reduces us to the following possibilities:

H_0	G_0
J_2	$G_2(4)$ or $F_4(2)$
Fi_{22}	${}^2E_6(2)$

If $(H_0, G_0) = (J_2, G_2(4))$ then G has rank 3 and degree 416 (see [22, p.97]), so has an even subdegree; and J_2 does not occur as the socle of a maximal subgroup of $F_4(2)$ or its automorphism group by [59]. For $(H_0, G_0) = (Fi_{22}, {}^2E_6(2))$, let T be a subgroup of H of order 11. Then $C_G(T) \not\leq H$ and $\langle T, S \rangle = H_0$ for any Sylow 2-subgroup S of H_0 , so Lemma 2.2 gives the conclusion.

Now suppose that H_0 contains no long root element of G . Let $u \in H_0$ be an element of order p . Then $|u^G \cap H|^2 > |u^G|$, so using [49] and the lower bound for $|u^G|$ in Lemma 7.3 (and also the known lists of maximal subgroups for G_0 of type ${}^2F_4, G_2, {}^3D_4, {}^2G_2, {}^2B_2$), we see that G_0, H_0 are as in the following table:

G_0	H_0	T
${}^2E_6(2)$	$Fi_{22}, \Omega_7(3), J_3, A_{12}$	$11, 13, [3^5], [3^5]$
$F_4(3)$	${}^3D_4(2)$	$[2^{12}]$
$F_4(2)$	$L_4(3), J_2, A_{10}$	$13, [3^3], [3^4]$
$G_2(4)$	$L_2(13)$	7
$G_2(3)$	$L_2(13)$	7

(Recall that we already eliminated $G_0 = {}^2F_4(2)'$ in Lemma 7.7 and the case $(G_0, H_0) = (G_2(4), J_2)$ was done in the previous paragraph.) For the remaining cases we choose a subgroup T of H_0 as in the table and apply Lemma 2.2. \square

7.3. Proof of $\frac{3}{2}$ -transitivity. Here we prove Theorem 7.2. Let G be an almost simple group of exceptional Lie type with socle $G_0 = G(q)$ ($q = p^a$), and suppose that G acts primitively on a set Ω with point stabiliser $H = G_\alpha$.

(A) Assume first that p divides $|\Omega|$. If p divides $|H|$ then by Theorem 7.1, G has a subdegree divisible by p , so cannot be $\frac{3}{2}$ -transitive. Now consider the case where H is a p' -group, that is, has order coprime to p . By Lemma 7.6 this means that one of the following holds:

- (i) H is a maximal torus normalizer (as in Lemma 7.6(ii));
- (ii) H is a maximal local subgroup (as in Lemma 7.6(v));
- (iii) $G_0 = E_8(q)$ and $H \cap G_0 = (\text{Alt}_5 \times \text{Alt}_6).2^2$ (as in Lemma 7.6(vi));
- (iv) $\text{soc}(H)$ is a simple group that is not a group of Lie type of characteristic p (as in Lemma 7.6(viii)).

Consider case (i). Here H is as in [46, Table 5.2]. The fact that p does not divide $|H|$ implies that G is of type E_8, E_7, E_6^ϵ or 3D_4 and $p \geq 5, 11, 5$ or 3 respectively. Moreover, one checks that there is a prime $r \leq 5$ which divides both $|H|$ and $|G : H|$. Let $s \in H$ be a (semisimple) element of order r . Then by [38, 4.2], we have $|s^G| > q^{112}, q^{53}, q^{31}$ or q^{16} respectively. A glance at [46, Table 5.2] shows that $|H|^2$ is much less than $|s^G|$. Hence by Lemma 2.6(ii), G has a subdegree divisible by r . Since r divides $|G : H| = |\Omega|$, it follows that G is not $\frac{3}{2}$ -transitive on Ω .

Now consider case (ii). Here H is as in [21, Theorem 1], so as H is a p' -group, one of the following possibilities holds:

G_0	$H \cap G_0$	p
$G_2(p)$	$2^3.L_3(2)$	$p = 5$ or $p \geq 11$
$F_4(p)$	$3^3.L_3(3)$	$p \geq 5$
$E_6^\epsilon(p)$	$[3^6].L_3(3)$	$p \geq 5$
$E_8(p^a)$	$[2^{15}].L_5(2)$	$p \geq 11, a = 1$
	$5^3.L_3(5)$	$p \geq 7, a \leq 2$

For G of type E_8 or E_6^ϵ we use the argument of the previous paragraph, taking $r = 2$. For $G = F_4(p)$ we also use this argument with $r = 2$, noting that $|s^G \cap H|$ is at most the number of involutions in H , which is 351, while $|s^G| > p^{16}$ by [38, 4.2]. Finally for $G = G_2(p)$, 3 divides $|H|$ and $|G : H|$, and H has 224 elements of order 3, so Lemma 2.6(ii) gives a subdegree divisible by 3 unless $p = 5$. When $p = 5$, G has base 2 [12, Table 12] and so is not $\frac{3}{2}$ -transitive.

Case (iii) is easily dealt with using the above argument with $r = 2$.

Finally consider case (iv). Here the possibilities for H_0 are given by [49] (and also the known lists of maximal subgroups for G_0 of type 2F_4 , G_2 , 3D_4 , 2G_2 , 2B_2). In all cases both $|H|$ and $|G : H|$ are divisible by 2. Taking an involution $s \in H$, it is easy to check that $|H|^2 < |s^G|$ with the following exceptions: $H_0 = {}^3D_4(2) < F_4(5)$ and $H_0 = U_3(3) < G_2(5)$. However in these exceptional cases one checks that $i_2(H)^2 < |s^G|$ (where $i_2(H)$ is the number of involutions in H). Hence 2.6(ii) shows that there is an even subdegree in all cases, and so G is not $\frac{3}{2}$ -transitive.

(B) Now assume that p does not divide $|\Omega|$. Then H is a parabolic subgroup. By Lemma 5.5, except in the cases where $G_0 = E_6(q)$ and $H = P_i$ ($i = 1, 3, 5, 6$), G has a unique nontrivial suborbit of size a power of p , and hence is not $\frac{3}{2}$ -transitive provided it is not 2-transitive (which does occur when $G_0 = {}^2B_2(q)$ or ${}^2G_2(q)$). Finally, consider the case $G_0 = E_6(q)$. We can take $H = P_1$ or P_3 (the others are images of these under the graph automorphism). The subdegrees of G on cosets of P_1 are given in [44], and are not equal.

It remains to consider the action of $E_6(q)$ with point stabiliser P_3 . Working in the algebraic group with the usual labelling of the root system, we have $P_3 = QL$ with unipotent radical Q and Levi subgroup L , where

$$Q = \left\langle U_\alpha \mid \alpha = \sum_{i=1}^6 c_i \alpha_i, c_3 > 0 \right\rangle, \quad L = \langle U_{\pm \alpha_i}, T \mid i = 1, 2, 4, 5, 6 \rangle,$$

where T is a maximal torus. Let $n_0 \in N_G(T)$ project to the longest element w_0 of the Weyl group $W(E_6)$. Recall that w_0 acts on the root system as the negative of the graph symmetry. We calculate the intersection $P_3 \cap P_3^{n_0}$ along the lines of [20, 2.8]. Observe that

$$\begin{aligned} L \cap L^{n_0} &= \langle U_{\pm \alpha_i}, T \mid i = 1, 2, 4, 6 \rangle, \\ Q \cap L^{n_0} &= \langle U_\alpha \mid \alpha = \alpha_3, \alpha_{34}, \alpha_{234}, \alpha_{13}, \alpha_{134}, \alpha_{1234} \rangle, \\ L \cap Q^{n_0} &= \langle U_\alpha \mid \alpha = -\alpha_5, -\alpha_{45}, -\alpha_{245}, -\alpha_{56}, -\alpha_{456}, -\alpha_{2456} \rangle, \\ Q \cap Q^{n_0} &= 1 \end{aligned}$$

where we use the notation $\alpha_{ij\dots} = \alpha_i + \alpha_j + \dots$. It follows that $P_3 \cap P_3^{n_0} = U_{12} \cdot (L \cap L^{n_0}) = U_{12} \cdot (A_1 A_1 A_2 T_2)$, where U_{12} is a unipotent group of dimension 12. Taking fixed points of a Frobenius morphism and returning to the finite group $G_0 = E_6(q)$, we see that in its action on P_3 , there is a subdegree equal to

$$\begin{aligned} |P_3 : P_3 \cap P_3^{n_0}| &= \frac{q^{36}(q^5 - 1)(q^4 - 1)(q^3 - 1)(q^2 - 1)^2(q - 1)}{q^{17}(q^3 - 1)(q^2 - 1)^2(q - 1)^2} \\ &= q^{19}(q^2 + 1)(q^5 - 1)/(q - 1). \end{aligned}$$

This subdegree does not divide $|G : P_3| - 1$, so this action is not $\frac{3}{2}$ -transitive.

This completes the proof of Theorem 7.2.

8. SPORADIC GROUPS

In this section we prove

Theorem 8.1. *Every almost simple $\frac{3}{2}$ -transitive primitive permutation group with socle a sporadic group, is doubly transitive.*

TABLE 15. Non-base two actions of large sporadic groups

G	point stabiliser H	$m = \text{hcf}(H , n - 1)$	Comment
Th	${}^3D_4(2).3$	1	
	$2^5.L_5(2)$	8	
Fi_{23}	$2.Fi_{22}$	—	rank 3, subdegrees 3510, 28160
	$P\Omega_8^+(3).S_3$	—	rank 3, subdegrees 28431, 109200
	$\Omega_7(3) \times S_3$	—	$O_3(H) \neq 1$
	$Sp_8(2)$	5	
	$3^{1+8}.2^{1+6}.3^{1+2}.2S_4$	3	
	$2^2.U_6(2).2$	$2^2 \cdot 7 \cdot 11$	
	$2^{11}.M_{23}$	46	
Co_1	Co_2	—	rank 4, subdegrees 4600, 46575, 47104
	$3.Suz.2$	11	
	Co_3	23	
	$U_6(2).S_3$	1	
	$(A_4 \times G_2(4)).2$	13	
	$2^{2+12}.(A_8 \times S_3)$	2	
	$2^{4+12}.(S_3 \times 3S_6)$	2	
	$2^{11}.M_{24}$	46	
	$2^{1+8}.\Omega_8^+(2)$	2	
	$2^{11}.M_{24}$	4	
	$2^{1+12}.3.M_{22}$	6	
	$2^{10}.L_5(2)$	1	
Fi'_{24}	Fi_{23}	—	rank 3, subdegrees 31671, 275264
	$2.Fi_{22}.2$	1	
	$(3 \times P\Omega_8^+(3).3).2$	1	
	$3^7.\Omega_7(3)$	1	
	$\Omega_{10}^-(2)$	1	
	$3^{1+10}.U_5(2).2$	3	
	$2^{11}.M_{24}$	46	
	$(2 \times 2^2.U_6(2)).S_3$	11	
Fi_{24}	$2.^2E_6(2).2$	—	$O_2(H) \neq 1$
BM	$(2^2 \times F_4(2)).2$	—	$O_2(H) \neq 1$
	$2^{9+16}.Sp_8(2)$	4	
	$2^{2+10+20}.(M_{22}.2 \times S_3)$	2	
	$2^{1+22}.Co_2$	46	
	Fi_{23}	3	
	Th	31	
M	$2.BM$	—	$O_2(H) \neq 1$

Let G be an almost simple primitive permutation group with socle a sporadic group L and point stabiliser H . The base two permutation representations of such groups G were determined in [13] and [58]. Such groups are not Frobenius groups and so are not $\frac{3}{2}$ -transitive. Hence we only need to consider those actions which are not base two. The non-base two actions of all nineteen sporadic groups of order up to $|Ly|$ and their automorphism groups are given in Table 16. We were able to compute all the subdegrees for these actions, and these are listed in Table 16. This gives much more information than we actually need, but might be interesting to the reader.

For the eight almost simple sporadic groups larger than Ly , the non-base two primitive actions are as in Table 15. In the third column of the table, with a few exceptions, we give the highest common factor m of the numbers $|H|$ and $n - 1$, where n is the degree $|G : H|$. If G is $\frac{3}{2}$ -transitive, the subdegree must divide m . When $G = Fi_{23}$ and $H = 3^{1+8}.2^{1+6}.3^{1+2}.2S_4$, it follows that the subdegree is 3. However, a simple MAGMA [7] calculation finds subdegrees of length greater than 3. In all other cases in the table, H is insoluble and so the subdegree must be at least 5. However, it is clear in all these cases that H has no transitive action of degree at least 5 and dividing m , hence G cannot be $\frac{3}{2}$ -transitive. In the exceptional cases in the table where m is not given, the fourth column either gives the subdegrees, or states that

$O_p(H) \neq 1$ for some prime p . In the latter cases we check that p does not divide $n - 1$, hence G is not $\frac{3}{2}$ -transitive.

This completes the proof of Theorem 8.1.

Table 16: Nontrivial subdegrees of some of the sporadic almost simple groups where the respective action does not have a base of size 2.

G	H	Subdegrees
M_{11}	M_{10}	10
	$L_2(11)$	11
	$M_9 : 2$	18, 36
	S_5	15, 20, 30
M_{12}	M_{11}	11
	$A_6.2^2$	20, 45
	$L_2(11)$	11^2 , 55, 66
	$3^2 : \text{AGL}_2(3)$	12, 27, 72, 108
	$2 \times S_5$	10^2 , 15, 30^2 , 60^3 , 120
	$2^{1+4} : S_3$	6, 16, 24, 32^2 , 48^2 , 96^3
	$4^2 : D_{12}$	6, 16, 24, 32^2 , 48^2 , 96^3
$M_{12}.2$	$3^{1+2} : D_8$	6, 18^2 , 27, 54^3 , 108^6
J_1	$L_2(11)$	11, 12, 110, 132
M_{22}	$L_3(4)$	21
	$2^4 : A_6$	16, 60
	A_7	70, 105
	$2^4 : S_5$	30, 40, 160
	$2^3 : L_3(2)$	7, 42, 112, 168
	M_{10}	30, 45, 180, 360
	$L_2(11)$	55^2 , 66, 165, 330
J_2	$U_3(3)$	36, 63
	$3.A_6.2$	36, 108, 135
	$2^{1+4} : A_5$	10, 32^2 , 80, 160
	$2^{2+4} : (3 \times S_3)$	12, 32, 96, 192^2
	$A_4 \times A_5$	15, 20, 24, 180, 240, 360
	$A_5 \times D_{10}$	12, 25, 50, 60^2 , 100^2 , 150^2 , 300
	$L_3(2) : 2 \times 2$	21, 28^2 , 42, 84^2 , 168^5 , 336^2
M_{23}	M_{22}	22
	$L_3(4) : 2$	42, 210
	$2^4 : A_7$	112, 140
	A_8	15, 210, 280
	M_{11}	65, 330, 792
	$2^4 : (3 \times A_5) : 2$	20, 60, 90, 160, 480^3
	M_{22}	22, 77
HS	$U_3(5) : 2$	175
	$L_3(4) : 2$	42, 105, 280, 672
	S_8	28, 105, 336, 630
	$2^4.S_6$	15, 32, 90, 120, 160, 192, 240, 240, 360, 960, 1440
	$4^3 : L_3(2)$	28, 64, 112, 336, 448, 896^2 , 1344
	M_{11}	55, 132, 165, 495, 660, 792, 1320, 1980
	$4.2^4.S_5$	30, 80, 128, 480, 640, 960, 1536, 1920
$HS.2$	$(2 \times A_6.2.2).2$	24, 30, 45, 72, 180, 288, 360^5 , 720^2 , 1440^4 , 2880^2
J_3	$L_2(16) : 2$	85, 120, 510, 680, 1360, 1360, 2040
$J_{3.2}$	$L_2(16) : 4$	85, 120, 510, 680, 2040, 2720
	$(3 \times M_{10}) : 2$	80, 135, 180, 540, 720, 1080^2 , 1440^3 , 2160^7
M_{24}	M_{23}	23
	$M_{22} : 2$	44, 231
	$2^4 : A_8$	30, 280, 448
	$M_{12} : 2$	495, 792
	$2^6 : 3.S_6$	90, 240, 1440
	$L_3(4) : S_3$	63, 210, 630, 1120
	$2^6 : (L_3(2) \times S_3)$	42, 56, 1008, 2688
	$L_2(23)$	253^2 , 276^4 , 506^2 , 759, 1012^5 , 1518^3 , 3036^9
	$U_4(3)$	112, 162
	M_{22}	330, 462, 1232
McL	$U_3(5)$	252, 750, 2625, 3500
	$3^{1+4} : 2.S_5$	90, 1215, 2430, 11664
	$3^4 : M_{10}$	30, 60, 162, 810, 1620^3 , 3645, 5832
	$L_3(4) : 2$	112, 210^3 , 1120, 1260, 2520^2 , 3360^3 , 4032
	$2.A_8$	210, 2240, 5040, 6720, 8064
	$2^4 : A_7$	112, 140, 210, 420, 672, 1680^2 , 2240, 3360^2 , 5040

<i>McL.2</i>	$M_{11} \times 2$	165, 220 ² , 660, 792 ² , 990, 1320, 1980 ⁴ , 3960 ⁴ , 5280, 7920 ⁹
<i>He</i>	$\text{Sp}_4(4) : 2$	136 ² , 425, 1360
	$2^2.L_3(4).S_3$	105, 720, 840 ² , 1344, 4480
	$2^6 : 3.S_6$	90, 120, 384, 960 ² , 1440, 2160, 2880 ² , 5760, 11520
<i>Ru</i>	${}^2F_4(2)$	1755, 2304
	$2^6.U_3(3).2$	63, 756, 2016 ³ , 16128 ² , 21504, 24192, 48384, 55296
	$(2^2 \times Sz(8)) : 3$	455, 3640, 5824, 29120 ² , 58240, 87360 ² , 116480
	$2^{3+8} : L_3(2)$	28, 672, 896, 2688, 3584, 4096, 10752, 14336 ² , 28672, 43008 ² , 57344, 86016, 114688
	$U_3(5) : 2$	126, 350, 2520, 5250, 7875, 10500, 12600 ² , 15750 ³ , 18000, 21000, 63000 ⁵ , 126000
<i>Suz</i>	$2^{1+4+6}.S_5$	30, 240, 480, 640, 3840 ² , 4096, 5120, 7680 ² , 10240, 12288, 15360, 30720, 61440 ⁶ , 122880
	$G_2(4)$	416, 1365
	$3.U_4(3).2$	280, 486, 8505, 13608
	$U_5(2)$	891, 1980, 2816, 6336, 20736
	$2^{1+6}.U_4(2)$	54, 360, 1728, 5120, 9216, 17280, 46080, 55296
	$3^5 : M_{11}$	165, 891, 2673 ² , 2916, 16038 ² , 17820, 40095 ³ , 53460
	$J_2 : 2$	200, 315, 630, 1800, 3150, 12600 ² , 16800, 20160, 25200 ² , 50400, 100800 ²
	$2^{4+6} : 3A_6$	60, 480, 1536, 1920, 6144 ² , 20480, 23040 ² , 46080, 92160, 184320
	$(A_4 \times L_3(4)) : 2$	224, 315, 420, 1260 ² , 1680, 2520, 15120, 20160 ² , 26880, 30240 ³ , 40320, 60480, 80640, 120960 ² , 161280 ²
	$2^{2+8} : (A_5 \times S_3)$	30, 40, 48, 480, 640 ² , 960 ² , 1536, 2048, 3072, 5120 ² , 7680 ³ , 10240 ² , 15360 ⁵ , 30720 ² , 61440 ⁴ , 92160 ³ , 122880 ⁴
<i>Suz.2</i>	$M_{12} : (2 \times 2)$	264, 495, 792, 990, 1760, 2640, 2970, 5280, 7920, 11880 ³ , 15840 ² , 19008, 31680 ³ , 47520 ¹⁰ , 63360, 95040 ⁵ , 190080 ⁶
<i>O'N</i>	$L_3(7) : 2$	5586, 6384, 52136, 58653
<i>O'N.2</i>	$4.L_3(4) : 2$	448, 630, 1120, 2240, 4480, 20160 ⁵ , 23040 ² , 40320 ³ , 80640 ⁴ , 161280 ¹⁴
<i>Co₃</i>	$McL : 2$	275
	HS	352, 1100, 4125, 5600
	$U_4(3).(2.2)$	224, 324, 1680, 4536 ² , 8505, 18144
	M_{23}	253, 506, 1771, 7590, 8855, 14168, 15456
	$3^5 : (2 \times M_{11})$	495, 2673, 32076, 40095, 53460
	$2.\text{Sp}_6(2)$	630, 1920, 8960, 30240, 48384, 80640
	$U_3(5) : S_3$	525, 2625, 3500, 6000, 21000, 23625, 31500, 63000 ² , 126000 ² , 189000
	$3^{1+4} : 4S_6$	180, 360, 3645, 7290, 14580, 29160, 58320 ⁴ , 69984, 349920
	$2^4.A_8$	70, 840, 896 ² , 960, 1120 ² , 1920 ² , 2688 ² , 4480 ⁴ , 6720 ² , 8960, 10080 ² , 13440 ⁷ , 17920, 26880 ⁶ , 32256, 40320 ⁴ , 53760 ⁸ , 80640 ³ , 161280 ²
<i>Co₂</i>	$U_6(2) : 2$	891, 1408
	$2^{10} : M_{22} : 2$	462, 2464, 21120, 22528
	McL	275, 2025, 7128, 15400, 22275
	$2^{1+8} : \text{Sp}_6(2)$	1008, 1260, 14336, 40320
	$HS : 2$	3850, 4125, 44352, 61600, 132000, 231000
	$(2^4 \times 2^{1+6}).A_8$	15, 210, 1680, 1920, 2520, 13440 ² , 20160, 35840, 161280, 344064, 430080
	$U_4(3) : D_8$	840, 1134, 1680, 8505, 9072, 19440, 181440 ³ , 204120 ² , 217728, 408240
	$2^{4+10}.(S_5 \times S_3)$	90, 120, 160, 480, 640, 720, 2880, 3840 ² , 5760 ² , 7680, 15360 ³ , 16384, 23040 ² , 40960 ² , 46080, 92160 ² , 184320, 245760 ² , 368640 ² , 737280, 983040
	M_{23}	506, 1771 ² , 5313, 7590, 15456, 17710, 28336 ² , 30360, 53130, 60720, 70840, 85008 ² , 141680, 170016, 212520 ⁴ , 283360, 425040 ² , 510048, 850080
<i>Fi₂₂</i>	$2.U_6(2)$	693, 2816
	$\Omega_7(3)$	3159, 10920
	$\Omega_8^+(2) : S_3$	1575, 22400, 37800
	$2^{10} : M_{22}$	154, 1024, 3696, 4928, 11264, 42240, 78848
	$2^6 : \text{Sp}_6(2)$	135, 1260, 2304, 8640, 10080, 45360, 143360, 241920 ²
	$(2 \times 2^{1+8}) : (U_4(2) : 2)$	270, 360, 1024, 1152, 4320, 34560, 40960, 46080, 69120 ² , 138240, 368640, 442368
	$U_4(3) : 2 \times S_3$	560, 1680, 1701, 2520, 17010, 68040, 81648, 90720 ² , 136080, 544320, 612360
	${}^2F_4(2)'$	1755, 11700, 14976, 83200 ² , 140400, 187200, 374400, 449280, 2246400
	$2^{5+8} : (S_3 \times A_6)$	48, 180, 480, 1536, 5760 ² , 7680, 8640, 11520, 24576 ² , 69120, 73728, 81920, 138240, 184320 ³ , 552960, 983040, 1105920
	$3^{1+6} : 2^{3+4} : 3^2 : 2$	Only some of the subdegrees 432, 1296, 2187, 5184, 8748, 10368, 34992, 46656, 69984, 139968, 279936, 314928, 419904, 559872, 839808, 1259712, 2519424
<i>Fi₂₂.2</i>	$3^5 : (2 \times U_4(2) : 2)$	Only some of the subdegrees 360, 486, 2916, 6561, 8640, 19440, 38880, 58320, 104976, 131220, 262440, 524880, 699840, 1049760, 1574640
	$G_2(3) : 2$	Only some of the subdegrees 702, 1456, 2808, 5824, 13104, 19656, 22113, 26208, 39312, 52416, 157248, 176904, 202176, 235872, 353808, 471744, 530712, 943488, 1415232, 2830464, 4245696
<i>HN</i>	A_{12}	462, 2520 ² , 10395, 16632, 30800, 69300, 166320 ² , 311850, 362880
	$2.HS.2$	1408, 2200, 5775, 35200, 123200, 277200, 354816, 739200
	$U_3(8) : 3$	1539, 14364, 25536 ³ , 68096, 131328, 229824 ² , 612864 ⁴ , 689472, 787968 ² , 5515776 ²
<i>Ly</i>	$G_2(5)$	19530, 968750, 2034375, 5812500
	$3.McL : 2$	15400, 534600, 1871100, 7185024

ACKNOWLEDGEMENTS

The authors thank an anonymous referee for a thorough reading of the manuscript.

REFERENCES

- [1] Seyed Hassan Alavi, *Triple factorisations of the general linear group by maximal parabolic subgroups*, preprint.
- [2] Seyed Hassan Alavi, John Bamberg, and Cheryl E. Praeger, *Triple factorisations of classical groups by maximal parabolic subgroups and their associated geometries*, preprint.
- [3] M. Aschbacher, *On the maximal subgroups of the finite classical groups*, Invent. Math. **76** (1984), no. 3, 469–514.
- [4] Michael Aschbacher and Gary M. Seitz, *Involutions in Chevalley groups over fields of even order*, Nagoya Math. J. **63** (1976), 1–91.
- [5] Eiichi Bannai, Shen Hao, and Sung-Yell Song, *Character tables of the association schemes of finite orthogonal groups acting on the nonisotropic points*, J. Combin. Theory Ser. A **54** (1990), no. 2, 164–200.
- [6] Áron Bereczky, *Maximal overgroups of Singer elements in classical groups*, J. Algebra **234** (2000), no. 1, 187–206.
- [7] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system I: The user language*, J. Symb. Comp. **24** 3/4 (1997), 235–265. Also see the Magma home page at <http://www.maths.usyd.edu.au:8000/u/magma/>.
- [8] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-regular graphs*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 18, Springer-Verlag, Berlin, 1989.
- [9] Timothy C. Burness, *Fixed point ratios in actions in finite classical groups. II*, J. Algebra **309** (2007), no. 1, 80–138.
- [10] Timothy C. Burness, Robert Guralnick, and Jan Saxl, *Base sizes for finite classical groups, I*, in preparation.
- [11] Timothy C. Burness, Robert M. Guralnick, and Jan Saxl, *On base sizes for symmetric groups*, Bull. London Math. Soc. **43** (2011), 386–391.
- [12] Timothy C. Burness, Martin W. Liebeck, and Aner Shalev, *Base sizes for simple groups and a conjecture of Cameron*, Proc. Lond. Math. Soc. (3) **98** (2009), no. 1, 116–162.
- [13] Timothy C. Burness, E.A. O’Brien, and Robert A. Wilson, *Base sizes for sporadic simple groups*, Israel J. Math. **177** (2010), 307–334.
- [14] W. Burnside, *Theory of groups of finite order*, Cambridge University Press, 1897.
- [15] Peter J. Cameron, *Finite permutation groups and finite simple groups*, Bull. London Math. Soc. **13** (1981), no. 1, 1–22.
- [16] ———, *Permutation groups*, London Mathematical Society Student Texts, vol. 45, Cambridge University Press, Cambridge, 1999.
- [17] A. R. Camina and J. P. J. McDermott, *On $\frac{3}{2}$ -transitive Frobenius regular groups*, J. London Math. Soc. (2) **20** (1979), no. 2, 205–214.
- [18] R. Carter, *Conjugacy classes in the Weyl group*, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Springer Lecture Notes 131, Springer, Berlin, 1970, pp. 297–318.
- [19] R. W. Carter, *Conjugacy classes in the Weyl group*, Compositio Math. **25** (1972), 1–59.
- [20] Roger W. Carter, *Finite groups of Lie type. conjugacy classes and complex characters*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1985.
- [21] Arjeh M. Cohen, Martin W. Liebeck, Jan Saxl, and Gary M. Seitz, *The local maximal subgroups of exceptional groups of Lie type, finite and algebraic*, Proc. London Math. Soc. (3) **64** (1992), no. 1, 21–48.
- [22] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985.
- [23] Charles W. Curtis, William M. Kantor, and Gary M. Seitz, *The 2-transitive permutation representations of the finite Chevalley groups*, Trans. Amer. Math. Soc. **218** (1976), 1–59.
- [24] John D. Dixon, *Permutation representations and rational irreducibility*, Bull. Austral. Math. Soc. **71** (2005), no. 3, 493–503.
- [25] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008.
- [26] Michael Giudici, Cai Heng Li, Cheryl E. Praeger, Ákos Seress, and Vladimir Trofimov, *On minimal subdegrees of finite primitive permutation groups*, Finite geometries, groups, and computation, Walter de Gruyter GmbH & Co. KG, Berlin, 2006, pp. 75–93.
- [27] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A. almost simple k-groups*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998.
- [28] Christoph Hering, *Transitive linear groups and linear groups which contain irreducible subgroups of prime order. II*, J. Algebra **93** (1985), no. 1, 151–164.
- [29] D. G. Higman and J. E. McLaughlin, *Geometric ABA-groups*, Illinois J. Math. **5** (1961), 382–397.
- [30] Derek F. Holt and Colva M. Roney-Douglass, *Constructing maximal subgroups of classical groups*, LMS J. Comput. Math. **8** (2005), 46–79 (electronic).
- [31] Robert B. Howlett, *On the degrees of Steinberg characters of Chevalley groups*, Math. Z. **135** (1973/74), 125–135.
- [32] N. Inglis, *Classical groups, multiplicity free representations and distance-transitive graphs*, Ph.D. thesis, Univ. of Cambridge, 1986.
- [33] J. P. James, *Two point stabilisers of partition actions of linear groups*, J. Algebra **297** (2006), no. 2, 453–469.
- [34] Peter Kleidman and Martin Liebeck, *The subgroup structure of the finite classical groups*, London Mathematical Society Lecture Note Series, vol. 129, Cambridge University Press, Cambridge, 1990.
- [35] Peter B. Kleidman, *The maximal subgroups of the finite 8-dimensional orthogonal groups $\mathrm{PO}_8^+(q)$ and of their automorphism groups*, J. Algebra **110** (1987), 173–242.
- [36] L. G. Kovács, *Primitive subgroups of wreath products in product action*, Proc. London Math. Soc. (3) **58** (1989), no. 2, 306–322.

- [37] R. Lawther and J. Saxl, *On the actions of finite groups of Lie type on the cosets of subfield subgroups and their twisted analogues*, Bull. London Math. Soc. **21** (1989), no. 5, 449–455.
- [38] Ross Lawther, Martin W. Liebeck, and Gary M. Seitz, *Fixed point ratios in actions of finite exceptional groups of Lie type*, Pacific J. Math. **205** (2002), no. 2, 393–464.
- [39] ———, *Fixed point spaces in actions of exceptional algebraic groups*, Pacific J. Math. **205** (2002), no. 2, 339–391.
- [40] Martin W. Liebeck, *The affine permutation groups of rank three*, Proc. London Math. Soc. (3) **54** (1987), no. 3, 477–516.
- [41] Martin W. Liebeck, Cheryl E. Praeger, and Jan Saxl, *On the 2-closures of finite permutation groups*, J. London Math. Soc. (2) **37** (1988), no. 2, 241–252.
- [42] ———, *Primitive permutation groups with a common suborbit, and edge-transitive graphs*, Proc. London Math. Soc. (3) **84** (2002), no. 2, 405–438.
- [43] Martin W. Liebeck and Jan Saxl, *The primitive permutation groups of odd degree*, J. London Math. Soc. (2) **31** (1985), no. 2, 250–264.
- [44] ———, *The finite primitive permutation groups of rank three*, Bull. London Math. Soc. **18** (1986), no. 2, 165–172.
- [45] Martin W. Liebeck, Jan Saxl, and Gary M. Seitz, *On the overgroups of irreducible subgroups of the finite classical groups*, Proc. London Math. Soc. (3) **55** (1987), no. 3, 507–537.
- [46] ———, *Subgroups of maximal rank in finite exceptional groups of Lie type*, Proc. London Math. Soc. (3) **65** (1992), no. 2, 297–325.
- [47] Martin W. Liebeck and Gary M. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and lie algebras*, AMS Maths Surveys and Monographs, to appear.
- [48] ———, *Subgroups generated by root elements in groups of Lie type*, Ann. of Math. (2) **139** (1994), no. 2, 293–361.
- [49] ———, *On finite subgroups of exceptional algebraic groups*, J. Reine Angew. Math. **515** (1999), 25–72.
- [50] ———, *A survey of maximal subgroups of exceptional groups of Lie type*, Groups, combinatorics & geometry (Durham, 2001), World Sci. Publ., River Edge, NJ, 2003, pp. 139–146.
- [51] Martin W. Liebeck and Aner Shalev, *Simple groups, permutation groups, and probability*, J. Amer. Math. Soc. **12** (1999), no. 2, 497–520.
- [52] E. Maillet, *Sur les isomorphes holoédriques et transitifs des groupes symétriques ou alternés*, J. Math. Pures Appl. **5** (1895), 5–34.
- [53] John P. J. McDermott, *Characterisations of some $\frac{3}{2}$ -transitive groups*, Math. Z. **120** (1971), 204–210.
- [54] Kenzo Mizuno, *The conjugate classes of Chevalley groups of type E_6* , J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), no. 3, 525–563.
- [55] ———, *The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8* , Tokyo J. Math. **3** (1980), no. 2, 391–461.
- [56] M. E. Muzychuk and I. N. Ponomarenko, *On pseudocyclic association schemes*, arXiv:0910.0682v1.
- [57] Peter M. Neumann and Cheryl E. Praeger, *An inequality for tactical configurations*, Bull. London Math. Soc. **28** (1996), no. 5, 471–475.
- [58] Max Neunhöffer, Felix Noeske, E.A. O’Brien, and Robert A. Wilson, *Orbit invariants and an application to the Baby Monster*, Journal of Algebra **341** (2011), no. 1, 297 – 305.
- [59] S. P. Norton and R. A. Wilson, *The maximal subgroups of $F_4(2)$ and its automorphism group*, Comm. Algebra **17** (1989), no. 11, 2809–2824.
- [60] D. S. Passman, *Solvable $3/2$ -transitive permutation groups*, J. Algebra **7** (1967), 192–207.
- [61] ———, *Solvable half-transitive automorphism groups*, J. Algebra **6** (1967), 285–304.
- [62] ———, *Exceptional $3/2$ -transitive permutation groups*, Pacific J. Math. **29** (1969), 669–713.
- [63] Jan Saxl, *On finite linear spaces with almost simple flag-transitive automorphism groups*, J. Combin. Theory Ser. A **100** (2002), no. 2, 322–348.
- [64] Gary M. Seitz, *Flag-transitive subgroups of Chevalley groups*, Ann. of Math. (2) **97** (1973), 27–56.
- [65] Ken-ichi Shinoda, *The conjugacy classes of Chevalley groups of type (F_4) over finite fields of characteristic 2*, J. Fac. Sci. Univ. Tokyo Sect. I A Math. **21** (1974), 133–159.
- [66] F. G. Timmesfeld, *Groups generated by root-involutions. I*, J. Algebra **33** (1975), 75–134.
- [67] Harold N. Ward, *On Ree’s series of simple groups*, Trans. Amer. Math. Soc. **121** (1966), 62–89.
- [68] Helmut Wielandt, *Finite permutation groups*, Translated from the German by R. Bercov, Academic Press, New York, 1964.
- [69] P.-H. Zieschang, *Theorems like Wielandt’s*, Arch. Math. (Basel) **58** (1992), no. 3, 209–213.

E-mail address: John.Bamberg@uwa.edu.au

E-mail address: Michael.Giudici@uwa.edu.au

E-mail address: m.liebeck@imperial.ac.uk

E-mail address: Cheryl.Praeger@uwa.edu.au

E-mail address: saxl@dpms.cam.ac.uk

(John Bamberg, Michael Giudici, Cheryl E. Praeger) CENTRE FOR THE MATHEMATICS OF SYMMETRY AND COMPUTATION, SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF WESTERN AUSTRALIA, 35 STIRLING HIGHWAY, CRAWLEY, WESTERN AUSTRALIA 6009

(Martin W. Liebeck) DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, LONDON SW7 2BZ, ENGLAND

(Jan Saxl) DPMMS, CMS, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE CB3 0WB, ENGLAND